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Base-controlled mechanical systems and geometric phases

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Abstract

In this paper, we carry a detailed study of mechanical systems with configuration space $Q \longrightarrow Q/G$ for which the base Q/G variables are being controlled. The overall system's motion is considered to be induced from the base one due to the presence of general non-holonomic constraints. It is shown that the solution can be factorized into dynamical and geometrical parts. Moreover, under favorable kinematical circumstances, the dynamical part admits a further factorization since it can be reconstructed from an intermediate (body) momentum solution, yielding a reconstruction phase formula. Finally, we apply this results to the study of concrete mechanical systems.

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1. Introduction

We shall describe a general formalism for studying classical mechanical systems in which some of the configuration degrees of freedom are being *controlled*, meaning that these are *known functions of time*. We will work under the (differential geometric-kinematical) assumption that the controlled variables live in the base of a principal fiber bundle $Q \longrightarrow Q/G = B$. The remaining variables can be thought of as living in a Lie group G and the equations for these *fiber unknowns* are derived by the hypothesis that the overall motion respects some (general) non-holonomic constraints which are present in the system.

A special case is that in which the underlying momentum map give conserved quantities, even when some of the variables are being acted by control forces. In this case, it is clear that motion in the base variables must induce motion in the remaining group variables in order for the momentum to be constant during the resultant motion. A concrete example is given by a self-deforming body for which the shape evolution (base variables) is known and global reorientation (group unknown) is induced by total angular momentum conservation [3].

In this paper, we consider the more general situation in which fiber motion is induced from the base one by the presence of (linear or affine) non-holonomic constraints. These are represented by a distribution $D \subset TQ$ [2,4] and we

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shall refer to them as *D-constraints*. The information telling us how the base variables are moving is represented by a base curve $\tilde{c}(t) \in B$ or, equivalently, by a curve $d_0(t) \in Q$ projecting onto $\tilde{c}$. The desired curve $c(t) = g(t) \cdot d_0(t) \in Q$ describing the full *system's physical motion* is defined by the requirement that it projects onto $\tilde{c}$ on the base at each time (i.e. the base variables are the given controlled ones) and that it satisfies the corresponding equations of *motion* plus the *D-constraints*. The base-controlled hypothesis can be seen as a set of *time-dependent constraints* and $g(t) \in G$ as the d_0-dependent (or *gauge*-dependent, see Section 2.4.3) fiber unknown.

The corresponding equations for $g(t)$ are derived by making *dynamical* assumptions, i.e. assumptions on the nature of the forces acting on the system. By using variational techniques, we give explicitly the equations of motion in Section 2. They correspond to the *non-holonomic momentum equation* of [2] with time-dependent coefficients evaluated along $d_0(t)$. Using the kinematical structure of the system, in Sections 2.4.3 and 4.1, we show how the solution $c(t)$ can be *factorized* by considering specific gauges $d_0(t)$, yielding that each factor has either a pure geometrical (kinematical) definition or it obeys dynamical equations which are simpler than the overall fiber ones.

In Section 3, we shall carry out a detailed analysis of systems with a special kinematical structure, focusing on the geometric-dynamical factorization of the solution mentioned above. Moreover, in Section 4, we show that under favorable kinematical circumstances (e.g. in the presence of *horizontal symmetries* [2]), the dynamical factor $g(t)$ of the solution $c(t) \in Q$ admits a further factorization. In fact, we can write reconstruction *phase formulas* [6] for $g(t)$. The obtained phase formulas relate the overall system's evolution to the geometry and dynamics of simpler intermediate solutions which, in turn, live in smaller spaces (coadjoint orbits). Consequently, these formulas generalize the ones obtained in [10,3] for rigid bodies and self-deforming bodies, respectively, to the more general setting of D-constrained induced motion. Notice that phase formulas become interesting and useful when the dynamical contribution can be expressed in terms of the system's dynamical quantities like energy and/or characteristic times (see, e.g., [10]). This is generically accomplished in Section 4 and exemplified in Section 5.

The formalism presented in this work, for studying *D-constrained, base-controlled* systems, applies to a larger class of mechanical systems than the one encoded in [3]. First, it applies to systems with general configuration space Q endowed with a principal bundle structure.[1] Also, in the second place, it allows for (linear or affine) D-constraints, and not only momentum conservation, to rule the system's dynamics. Indeed, in examples 5.3 and 5.4, we are able to answer two natural questions which arise from [3]: what happens to the corresponding *phase formulas* when magnetic-type forces are acting upon a deforming body, and thus, when the (angular) momentum is no longer conserved?; how does a self-deforming body move when there are additional (internal) non-holonomic constraints between the (no longer controllable) shape variables?

To end this introduction, we would like to comment on the applications of the present work to mechanical control theory. First, note that control problems are, in a sense, *orthogonal* to the one we described so far. In this paper, we claim to *know* the base variables dynamics and we want to *find* the induced fiber motion; while in control theory one *starts* with a desired fiber dynamics and tries to *find* which base curve induces it (and, after that, how to implement this base motion via control forces). Nevertheless, the spirit of this paper is to think of the known base dynamics as coming from direct observation or measurement (example 5.3 illustrates this point very clearly). Indeed, an interesting feature of this kind of systems is the fact that the overall motion $c(t)$ can be constructed from that in the base using only the *kinematics* of $\tilde{c}(t)$ and without actually knowing the *forces* which are inducing such base motion. The results on the induced fiber motion, obtained by the formalism we describe below, can be thus used for theoretically correcting the a priori fiber dynamics prediction when the observed base dynamics deviates from the control-theoretical desired one. Also, analytical phase formulas provide interesting tools for directly testing different control configurations and theoretical methods.

2. Controlled systems with additional non-holonomic constraints

2.1. The kinematical setting

In the remaining, we shall focus on mechanical systems with general non-holonomic linear (or affine, see Section 3.2) constraints. More precisely, our setting consists of a mechanical system described by the data (Q, L, G, D):

[1] Notice that in [3], Q represented specifically the configuration space of a deforming body.

- Let Q denote the *configuration space* and G a symmetry Lie group acting on Q by the left such that $Q \xrightarrow{\pi} Q/G$ is a *principal* G*-bundle*. We shall call, as usual, $B := Q/G$ the *shape space* (see [9]). We denote the action by $g \cdot q$ and the induced *infinitesimal action* by $\rho_{g*} : TQ \longrightarrow TQ$.
- Let $k_q(\cdot,\cdot)$ denote a G*-invariant Riemannian metric* on Q and $k_q(\cdot) : T_qQ \longrightarrow T_q^*Q$ the induced bundle isomorphism.
- Let $L : TQ \longrightarrow \mathbb{R}$ denote the G*-invariant Lagrangian* (with respect to the lifted G-action on TQ) given by the (k_q-)*kinetic energy* $K(\dot{q})$ minus G-invariant potentials (see also Appendix A).
- Let $D \subseteq TQ$ be a *constraint distribution*.

We shall assume further:

($H1$) D is G*-invariant* and $D_q + V_q = T_qQ$, for all $q \in Q$ and $\mathrm{Ver}_q = \mathrm{Ker}(\pi_{*q})$ denoting the vertical subspace of T_qQ. This is referred to as the *principal case* in [2].

Now, suppose that, for such a system, the *base variables are being controlled in a certain known way*. This means, that

($H2$) we are given a curve $d_0(t)$ in Q for $t \in I := [t_1, t_2]$ or, equivalently, a map $\tilde{c} : [t_1, t_2] \longrightarrow Q/G$ s.t. $\pi(d_0(t)) = \tilde{c}(t)$. The *time evolution of the controlled system* is then described by a curve $c(t) \in Q$ such that

$$\pi(c(t)) = \tilde{c}(t)$$

for each $t \in [t_1, t_2]$.

The above means that

$$c(t) = g(t) \cdot d_0(t) \tag{1}$$

where the curve $g(t)$ in G represents the ($d_0(t)$-dependent) *unknown* of our *controlled mechanical problem*.

Definition. We shall refer to the data $(Q, L, G, D, \tilde{c})$ as a *base-controlled* (D-)*constrained dynamical system*.

Note that, if the controlled problem has a unique solution $c(t)$ for each initial value $(c(t_1), \dot{c}(t_1)) \in T_{c(t_1)}\pi^{-1}(\tilde{c}(t_1))$, then for each curve $d_0(t)$ in Q lying over $\tilde{c}(t) \in Q/G$, there is a unique $g(t)$ satisfying (1). In this case, the initial values for the unknown in G read

$$\begin{aligned} c(t_1) &= g(t_1) \cdot d_0(t_1) \\ \dot{c}(t_1) &= \frac{\mathrm{d}}{\mathrm{d}t}(g \cdot d_0)(t_1). \end{aligned} \tag{2}$$

The curve $d_0(t)$ will be called *gauge choice* or, simply, *gauge*. This terminology is motivated by the analogy between the freedom in choosing among such curves projecting to the same $\tilde{c}$ in shape space and gauge freedom in classical gauge field theories (see [9,11], the references therein and also Section 2.4.2).

Remark 2.1 (*Restricted Configuration Space*). Note that ($H2$) implies (but it is not equivalent to!) the following *holonomic constraint*:

$$c(t) \in \pi^{-1}(\tilde{c}(I)).$$

For a specific problem in which $\tilde{c}$ is fixed, one can restrict the analysis to $\tilde{Q} = \pi^{-1}(\tilde{c}(I))$. Nevertheless, in what follows, we continue with the study of generic $\tilde{c}$'s and thus express the results in terms of the kinematical structure of the whole Q. Notice that this is the more convenient procedure for studying systems in which $\tilde{c}$ can be (dynamically) perturbed.

Remark 2.2 (*Vertical* D*-Constraints*). Note that the dimension assumption ($H1$) states that the D-constraints are *vertical*, in the sense that it ensures that the equations of motion (locally) drop to the base Q/G with no D-constraints remaining there. In other words, the base curve $\tilde{c}(t)$ can be arbitrarily chosen within Q/G. For example, if the sum is direct, i.e., $D_q \oplus V_q = T_qQ$ then D defines a principal connection and we are in the *purely kinematical* case [2] described in Section 4.3. In the case $D_q \cap V_q = 0$ but $D_q \oplus V_q \neq T_qQ$, then constraints are also to be considered in the motion of the base variables and, thus, the base dynamics could not be (arbitrarily) controllable.

2.1.1. Kinematical ingredients

We now recall some known definitions and properties for mechanical systems that we shall use through the paper.

First, recall that for simple mechanical systems with symmetry [1,7] as described above, the lifted G action on TQ always has an (equivariant) momentum map $J : TQ \longrightarrow \mathfrak{g}^*$ given by

$$\langle J(v_q), X\rangle = \langle k_q(v_q), X_Q(q)\rangle,$$

for $X \in \mathfrak{g}$. Let us also recall another ingredients (see e.g. [9]):

- *Locked inertia tensor* $I_q : \mathfrak{g} \longrightarrow \mathfrak{g}^*$,

  $$I_q = \sigma_q^* \circ k_q \circ \sigma_q$$

  with $\sigma_q : \mathfrak{g} \longrightarrow T_q Q$ denoting the *infinitesimal generator map*, $\sigma_q(X) = X_Q(q)$. I_q defines a symmetric, non-degenerate inner product in $\mathfrak{g}$. Because the metric k_q is G-invariant, I also satisfies the *equivariance* property:

  $$I_{g\cdot q} = Ad_g^* \circ I_q \circ Ad_{g^{-1}}.$$

- The momentum map J is Ad^*-equivariant, i.e.,

  $$J(g \cdot m) = Ad_g^* J(m)$$

  with $Ad_g^* = (Ad_{g^{-1}})^t$ denoting the (left) *coadjoint representation* of G on $\mathfrak{g}^*$ and ^{t} the transpose. This follows from the identity

  $$\sigma_{g\cdot q}(X) = \rho_{g_* q}(\sigma_q(Ad_{g^{-1}} X)).$$

Now, from (1) we have that

$$\frac{\mathrm{d}}{\mathrm{d}t} c(t) = \rho_{g(t)_* d_0(t)} \left(\sigma_{d_0(t)} \left(g^{-1} \frac{\mathrm{d}}{\mathrm{d}t} g(t)\right)\right) + \rho_{g(t)_* d_0(t)} \frac{\mathrm{d}}{\mathrm{d}t} d_0(t) \tag{3}$$

and thus,

$$J\left(\frac{\mathrm{d}}{\mathrm{d}t} c(t)\right) = Ad^*_{g(t)} I_{d_0(t)} \left(g^{-1} \frac{\mathrm{d}}{\mathrm{d}t} g(t)\right) + Ad^*_{g(t)} J\left(\frac{\mathrm{d}}{\mathrm{d}t} d_0(t)\right). \tag{4}$$

We can think of $J_0(t) := J(\frac{\mathrm{d}}{\mathrm{d}t} d_0(t))$ as the *apparent* or *internal* momentum along $d_0(t)$ and $I_0(t) := I_{d_0(t)}$ as the locked inertia tensor changing with the gauge motion $d_0(t)$. Notice that $J(\frac{\mathrm{d}}{\mathrm{d}t} c(t)) = Ad^*_{g(t)} \Pi(t)$ for

$$\Pi(t) := I(d_0(t))(g^{-1} \dot{g}(t)) + J(\dot{d_0})(t). \tag{5}$$

This $\Pi(t)$ represents the *body momentum*, i.e. the momentum $J(\dot{c}(t))$ as seen from a reference frame moving with $g(t)$ which, in turn, depends on the gauge choice $d_0(t)$.

2.2. Dynamical hypothesis

The assumption $(H2)$ above can be interpreted as giving a *time-dependent type of kinematical constraint* on the original system, in addition to the one represented by the distribution $D \subseteq TQ$. To determine the motion of such a *twice kinematically constrained system*, i.e. to find[2] $c(t)$ in Q, we need to add *dynamical information*. This information consists in assumptions about the nature of the *forces* which are acting upon the system in order to satisfy the imposed kinematical constraints.

For the set of constraints corresponding to the distribution D, we shall assume

$(DH1)$ *D' Alambert's Principle*: The *D-constraint forces* lie in the annihilator space of the kinematical distribution D.

2 Equivalently, for a chosen gauge $d_0(t)$, to find the corresponding $g(t)$ in G.

Denoting $F^D : TQ \longrightarrow \mathbb{R}$ the D-forces (seen as 1-forms on Q) which act on the system enforcing the D-constraints, $(DH1)$ means that

$$F^D(v) = 0$$

for all "virtual displacement" $v \in D \subset TQ$. If $(DH1)$ is not satisfied by the system's forces, we must then know[3] the D-constraint forces and add them to the equations of motion (see Section 2.4.1).

For the time-dependent *control like* constraints represented by the shape space curve $\tilde{c}(t) \in Q/G$, the assumption takes a less usual form:

$(DH2)$ The forces which are inducing the motion $c(t)$ to satisfy $\pi(c(t)) = \tilde{c}(t)$ are of a kind that we shall denote as *good internal* ones. Good internal forces seen as 1-forms $F^c_{\text{int}} : TQ \longrightarrow \mathbb{R}$ satisfy

$$F^c_{\text{int}}(\delta c) = 0$$

for all *vertical variations* $\delta c = \frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}(g(t,s) \cdot d_0(t))$ in D and any gauge $d_0(t)$ (see also below).

In other words, good internal forces are such that they do not affect dynamically (i.e. by adding extra terms) the *vertical* part of the equations corresponding to (Q, L, G, D). This idea is already present in [2], in terms of the validity of the non-holonomic momentum equations when internal (shape space) control forces are present.

Example 2.3 (*Motion of Self-Deforming Bodies*). Let $Q = \mathbb{R}^{3N-3}$ be the configuration space of an N-particles system modeling a deforming body. In this case, usual internal forces between the particles of the system satisfying the *strong action–reaction principle* [5] are *good internal forces*. For details, see [3].

Remark 2.4 (*Non-Food Internal Forces*). If the constraint forces acting on the controlled base variables are not of *good internal* type, then we must add the extra piece of information missing, this is, how the equations have to be modified by adding the non-vanishing terms $F^c(\delta c)$ (see 2.4.1). In the case of the above example, this means that if there are, say, electromagnetic forces acting on the self-deforming body which do not satisfy the *strong* action–reaction principle, then one must know the underlying magnetic field data and correct the angular momentum conservation equations as usual (see e.g. [5], and also Sections 4.5 and 5.4).

For control purposes, the equations of motion following from $(DH1)$ and $(DH2)$ for the base variables $r \in B = Q/G$ can be locally written as (for details see [2])

$$M(r)\ddot{r} = -C(r, \dot{r}) + N(r, \dot{r}, J(g, \dot{g})) + F^{\text{pot}} + F^c_{\text{int}}$$

where g denotes the (local) vertical part of variables in $Q \simeq Q/G \times G$, J the (generalized, non-holonomic) momentum, F^{pot} the potential forces acting on B and F^c_{int} the control forces mentioned in $(DH2)$. Also, M denotes the mass matrix of the system, C the Coriolis term (quadratic in $\dot{r}$) and N a term being quadratic in $\dot{r}$ and $\langle J, \xi \rangle$, where ξ is a q-dependent element in $\mathfrak{g} = \text{Lie}(G)$.

In what follows, we shall assume that the system is being base-controlled, so the control forces are inducing via the above equation the prescribed motion $\tilde{c}(t) \equiv r(t)$. The problem is then to find the remaining vertical part of the motion, which is induced by the one in the base B because of the presence of the D-constraints.

2.3. The variational principle

The equations of motion for the above described base-controlled D-constrained system, satisfying $(H1, 2)$ and $(DH1, 2)$, can be deduced from an adapted *variational principle*.

Explicitly, we shall assume that the solution curve $c(t)$ is an *extremal* of the action functional

$$S_Q = \int_{t_1}^{t_2} L(\dot{c})\mathrm{d}t$$

[3] Or to know some other information about them leading to the corresponding equations of motion.

for deformations of the following specific kind:

$$c(t, s) = g(s) \cdot c(t). \tag{6}$$

These kind of deformations can be called *vertical* following the ideas of [4]. Also following [4], from $(DH1)$ we shall restrict the variations to the ones *satisfying the D-constraints*, i.e., $\delta c \in D_{c(t)}$.

Let $I = [t_1, t_2]$ and $\Omega(Q; \tilde{c}(t), q_1, q_2)$ denote the space of smooth curves $I \longrightarrow Q$ with fixed end points $q_1, q_2 \in Q$ such that $\pi(c(t)) = \tilde{c}(t)$. Note that, for a given (any) gauge choice $d_0(t)$ such that $\pi(d_0(t)) = \tilde{c}(t)$, any deformation can be written as

$$c(t, s) = g(t, s) \cdot d_0(t). \tag{7}$$

and thus

$$\Omega(Q; \tilde{c}(t), q_1, q_2) \approx \Omega_{d_0}(G; g_1, g_2)$$

with $\Omega_{d_0}(G; g_1, g_2)$ being the space of smooth curves $I \longrightarrow Q$ with fixed end points g_i, s.t. $g_i \cdot d_0(t_i) = q_i$ for $i = 1, 2$.

So, summing up, our problem is equivalent to the following (gauge invariant) variational formulation:

$P1$ *(Gauge invariant formulation)*. Finding an extremal $c(t)$ of the action S_Q, i.e. $\delta S_Q = 0$, among the curves in $\Omega(Q; \tilde{c}(t), q_1, q_2)$ for vertical deformations $\delta c(t) = \frac{d}{ds}|_{s=0} c(t, s)$ induced by (6), vanishing at the end points, i.e. $\delta c(t_i) = 0$ for $i = 1, 2$, and with both $\dot{c}(t)$ and δc satisfying the D-constraints.

Once the gauge is fixed, the action S induces an equivalent *non-autonomous Lagrangian system* on the G which is, in turn, equivalent to the following:

$P2$ *(Gauge covariant formulation)*. Finding extremal curve $g(t)$ in the set $\Omega_{d_0}(G; g_1, g_2)$ for the action

$$S_G[d_0] = \int_{t_1}^{t_2} L_{d_0}(g, \dot{g}, t) dt,$$

i.e. $\delta S_G[d_0] = 0$, satisfying the gauge-induced D-constraints, i.e.,

$$\dot{d}_0(t) + (g^{-1}\dot{g})_Q(d_0(t)) \in D_{d_0(t)} \tag{8}$$

and for variations $\delta g(t) = \frac{d}{ds}|_{s=0} g(t, s)$, $\delta g(t_i) = 0$, $i = 1, 2$, satisfying the D-constraints:

$$\left(g^{-1} \left. \frac{d}{ds} \right|_{s=0} g \right)_Q (d_0(t)) \in D_{d_0(t)}. \tag{9}$$

Notice that, although different gauge choices shall lead to different time dependence of the $g(t)$ equation's coefficients, the full solution $c(t)$ is the same for all d_0's. In other words, though the equations for $g(t)$ (and thus $g(t)$ itself) are not *gauge invariant*, the solution $c(t)$ is. On the other hand, $g(t)$ can be seen as being *gauge covariant* (see Remark 2.5).

In the above formulation $(P2)$, L_{d_0} is $L(\dot{c})$ with $c(t)$ given by (1). It is easy to see that it can be put in the form of the (left) G-invariant non-autonomous Lagrangian given by Eq. (57) of Appendix A in terms of the body velocity $\xi = g^{-1}\dot{g}$.

Finally, note that variations $\delta\xi = \frac{d}{ds}|_{s=0}(g^{-1}\dot{g})$ induced by variations $\delta g = \frac{d}{ds}|_{s=0} g(t, s)$ satisfy the following identity:

$$\delta\xi - \frac{d}{dt}\left(g^{-1}\delta g\right) = [\xi, g^{-1}\delta g], \tag{10}$$

where $[,]$ denotes the Lie bracket on $\mathfrak{g}$.

Remark 2.5 (*Gauge Covariance*). Since D is G-invariant, (8) is *gauge covariant*: if $\tilde{d}_0(t) = g_{cg}(t) \cdot d_0(t)$ is another gauge, then $g(t)$ in (1) satisfies the D-constraint equation (8) for $d_0(t)$ iff $\tilde{g}(t) := g(t) g_{cg}^{-1}(t)$ satisfies the equation analogous to Eq. (8) for the new gauge $\tilde{d}_0(t)$.

2.4. Equations of motion

Note that, as a consequence of Newton's second law, the equations for the unknown $g(t)$ shall be *second-order* ones. Also, by the time-dependent control constraint, they shall also be *non-autonomous* and *gauge-dependent*, i.e., its coefficients will depend on time through the chosen $d_0(t)$.

We shall start with the *gauge invariant formulation* $(P1)$. Taking into account that the variations are of the form (6),

$$\delta S_Q = 0$$

straightforwardly implies

$$i^*_{c(t)}\left(\frac{\mathrm{d}}{\mathrm{d}t}J(\dot{c})\right) = 0 \tag{11}$$

where, for any $q \in Q$, $i_q : \mathfrak{g}^q \hookrightarrow \mathfrak{g}$ denotes the inclusion, $i^*_q : \mathfrak{g}^* \hookrightarrow (\mathfrak{g}^q)^*$ the canonical projection and

$$\mathfrak{g}^q := \{X \in \mathfrak{g},\ X_Q(q) \in D_q\}.$$

The above equation is equivalent to the *non-holonomic momentum equation* of [2], evaluated on the controlled curve $c(t)$ of Eq. (1).

Remark 2.6 (*Non-Necessity of* $(H1)$ *nor* $(H2)$). Eq. (11) is one of the equations of motion of any system whose kinematics is as in Section 2.1 without the need of $(H1, 2)$. The only dynamical hypothesis needed is $(DH1)$ plus the fact that any other force acting on the system (seen as 1-forms on Q) is such that it vanishes when evaluated on vertical variations. What these last kinematical hypothesis $(H1, 2)$ add is: that no D-constraints remain on the base variables and that these are being controlled, so (11) is the only *equation of motion* (not of constraint) left to solve in the system.

These are $k := \dim \mathfrak{g}^{c(t)} = \dim \mathfrak{g}^{c(t_1)} = \text{const.}$ equations coupled to the $(\dim \mathfrak{g} - k)$ number of D-constraint equations:

$$\dot{c}(t) \in D_{c(t)}.$$

Since the base control hypothesis $(H2)$ leave only $\dim \mathfrak{g}$ degrees of freedom, Eq. (11) and the above D-constraint equations determine uniquely $c(t)$ because of $(H1)$.

Below, we shall give more explicit equations for the unknown $g(t)$ by fixing a gauge choice $d_0(t)$ and working in the gauge covariant formulation $(P2)$.[4] To illustrate on the underlying calculation, we shall derive the equations directly from $(P2)$, though they can be also derived from (11) using the decomposition (1). Let us, thus, evaluate

$$0 = \delta S_G[d_0] = \int_{t_1}^{t_2} \left\langle I(d_0(t))\xi + J(\dot{d_0})(t), \delta\xi \right\rangle.$$

By Eq. (10) and integration by parts, we have

$$= -\int_{t_1}^{t_2} \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\left(I(d_0(t))\xi + J(\dot{d_0})(t)\right) + ad^*_\xi\left(I(d_0(t))\xi + J(\dot{d_0})(t)\right), g^{-1}\delta g\right\rangle$$

where $ad^*_\xi = -(ad_\xi)^t$ denotes the (left) coadjoint action. Notice that $g^{-1}\delta g$ is arbitrary only among variations satisfying the D-constraint (9), then

$$i^*_{d_0(t)}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(I(d_0(t))\xi + J(\dot{d_0})(t)\right) + ad^*_\xi\left(I(d_0(t))\xi + J(\dot{d_0})(t)\right)\right] = 0 \tag{12}$$

must hold. These are $k = \dim \mathfrak{g}^q$ (constant $\forall q \in Q$) equations of motion for the *body velocity* $\xi(t) = g^{-1}\dot{g}(t)$ which are coupled to the $(\dim \mathfrak{g} - k)$ *non-holonomic constraint equations* Eq. (8) also for $\xi(t)$.

[4] We do need $(H1, 2)$ for $(P2)$.

Before passing to the next section, we give some properties of the subspaces which are involved in (12) and which follow from the G-invariance of D.

Proposition 2.7. *The following holds:*
- $\mathfrak{g}^{g\cdot q} = Ad_g \mathfrak{g}^q$
- $Ad_g \circ i_q = i_{g\cdot q} \circ Ad_g$.

Example 2.8 (*The Purely Kinematical Case of [2]*). In this case, $D \cap T\mathrm{Orb}_G(Q)$ is trivial and thus $\mathfrak{g}^q = \{0\}$ for all $q \in Q$. Equation of motion (11) is trivial and the motion of the system is only determined by the constraint equation $\dot{c} \in D$. See also Section 4.3.

Example 2.9 (*The Case of Full Horizontal Symmetries [2]*). In this case, there exists a subgroup $H \subset G$ such that $\mathfrak{g}^q$ is constantly $\mathfrak{h} = \mathrm{Lie}(H)$ for all $q \in Q$. Then, $i^*_{\mathfrak{h}}(J(\dot{c}))$ is a conserved quantity along the solution $c(t)$. See also Section 4.4.

Example 2.10 (*The Case $D = TQ$ and Momentum Conservation*). When $D = TQ$ and so the D-constraints are trivial, equations (11) (equivalently, (12)) imply the conservation of the momentum J along the solution $c(t)$. This is the case, for example, of a self-deforming body which freely rotates around its center of mass with conserved angular momentum [3,11].

2.4.1. Applied forces

In the presence of arbitrary additional external forces $F : TQ \longrightarrow \mathbb{R}$, the corresponding equations of motion are

$$i^*_{c(t)}\left(\frac{\mathrm{d}}{\mathrm{d}t}J(\dot{c})\right) = i^*_{c(t)} \circ \sigma^*_{c(t)}(F_{c(t)}).$$

Also, Eq. (11) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}J(\dot{c}) = \Gamma_c(t)$$

for a curve $\Gamma_c(t) \in \mathrm{Ker}(i^*_{c(t)})$. This $\Gamma_c(t)$ can be interpreted as an *external (generalized) torque* caused by the forces F (see example 5.4). Within the gauge covariant formulation, the corresponding equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(I(d_0(t))\xi + J(\dot{d_0})(t)\right) + ad^*_\xi\left(I(d_0(t))\xi + J(\dot{d_0})(t)\right) = \sigma^*_{d_0(t)}\left(\rho^*_{g*d_0(t)}F_{c(t)}\right) =: \Gamma_{d_0}(t). \tag{13}$$

Now, assuming that there are no other forces than the D-constraint ones F^D and that $(DH1, 2)$ hold, we arrive at Eq. (13) with $F = F^D$. Notice that, since $(DH1)$ holds, Eq. (12) above follows by projecting via $i^*_{d_0(t)} : \mathfrak{g}^* \hookrightarrow (\mathfrak{g}^{d_0(t)})^*$. If we choose a splitting $\mathfrak{g} = \mathfrak{g}^{d_0(t)} \oplus \mathfrak{D}^{d_0(t)}$ with $P^{\mathfrak{D}} : \mathfrak{g} \longrightarrow \mathfrak{g}^{d_0(t)}$ the corresponding projector, we get that the *external torque* $\Gamma_{d_0}(t) \in \mathrm{Ker}i^*_{d_0(t)}$ present in the r.h.s. of Eq. (13) can be also written as[5]

$$\Gamma_{d_0}(t) = \left(1 - P^{\mathfrak{D}}\right)^* \circ \sigma^*_{d_0(t)}\left(\rho^*_{g*d_0(t)}F^D_{c(t)}\right).$$

Expression (13) gives $\dim \mathfrak{g}$ equations coupled to the $(\dim \mathfrak{g} - k)$ equations of D-constraints. Nevertheless, notice that in (13) we have $(\dim \mathfrak{g} - k)$ new unknowns: the D-constraint forces F^D.

2.4.2. Bundle formulation

The gauge invariant formulation $(P1)$ and the gauge fixed formulation $(P2)$ of the problem, both have as underlying G-bundle $Q_I \longrightarrow I$ which is related to $Q \longrightarrow Q/G$ by the pull-back diagram

$$\begin{array}{ccc} Q_I & \longrightarrow & Q \\ \downarrow & & \downarrow \\ I & \xrightarrow{\tilde{c}} & Q/G. \end{array}$$

[5] The following expression yields the same $\Gamma_{d_0}(t)$ for any choice of $P^{\mathfrak{D}}$ since $F_{c(t)}$ vanishes on $D \subset TQ$ by $(DH1)$.

Moreover, Q_I is a trivial G-bundle and the corresponding global sections are the *gauge curves* $d_0(t)$ projecting to $\tilde{c}(t)$ on shape space. Choosing a section, so $Q_I \approx I \times G$, we arrive at the non-autonomous system on G as described by $(P2)$.

Remark 2.11 (*Relation to 1-d Gauge Field Theories*). The setting above gives a description of our time-dependent problem in terms of a 1-dimensional *gauge field theory*. Here, the *fields* are the sections $I \longrightarrow Q_I \approx I \times G$ and G is the gauge group. See also [9] and the references therein. Notice that, in this context, the corresponding field theory is not gauge invariant since, actually, the problem consists in finding the correct gauge transformation taking d_0 into the desired solution $c = g \cdot d_0$.

There is also another set of bundles which are relevant for this problem, specially for the study of the equations of motion. These are the vector bundles $\mathfrak{g}^D$, $\mathfrak{g}^D_I$, which are related by the pull-back diagram

$$\begin{array}{ccc} \mathfrak{g}^D_I & \longrightarrow & \mathfrak{g}^D \\ \downarrow & & \downarrow \\ I & \xrightarrow{\tilde{c}} & Q \end{array}$$

with $\mathfrak{g}^D := \sqcup_{q\in Q} \mathfrak{g}^q$. The vector bundle $\mathfrak{g}^D$ can be also defined as $\sigma^{-1}(D)$, for the vector bundle morphism

$$\begin{aligned} \sigma &: Q \times \mathfrak{g} \longrightarrow TQ \\ &: (q, X) \longmapsto X_Q(q) \end{aligned}$$

with $D \subseteq TQ$ seen as a vector subbundle. Note that bundle $\mathfrak{g}^D_I$ is also trivial since I is contractible. For a given choice of gauge curve $d_0(t)$, there must exist a smooth curve $T(t) \in GL(\mathfrak{g})$ such that the set

$$\{T(t)X_i\}_{i=1}^{\dim \mathfrak{g}^{d(t_1)}} \tag{14}$$

is a basis of $\mathfrak{g}^{d_0(t)}$ if $\{X_i\}_{i=1}^{\dim \mathfrak{g}^{d(t_1)}}$ is a basis of the vector space $\mathfrak{g}^{d_0(t_1)} \subseteq \mathfrak{g}$. This is the pull-back (to $\mathfrak{g}^D_I$) version of the moving basis formulation of [2,4].

Remark 2.12 (*Vector Bundle Non-Triviality*). From Example 2.9 we see that the geometry of the bundle $\mathfrak{g}^D$ plays a crucial role in the form of the equations of motion. In other words, the geometry of $\mathfrak{g}^D$ enters in the *non-commutativity* of $\frac{\mathrm{d}}{\mathrm{d}t}$ and $i^*_{c(t)}$ in Eq. (11). Even though the bundle $\mathfrak{g}^D_I$ is always trivializable, if it is not directly *trivial*, the need of using time-dependent sections $T(t)$ enters non-trivially in the equations of motion. See also Sections 3.4 and 3.3 where this effect is isolated from others.

2.4.3. Non-holonomic gauges

Recall the constraint equations (8) which are coupled to the motion ones (12). Being explicitly *gauge-dependent*, a natural question that follows is: is there a gauge, i.e. a choice of $d_0(t)$, which simplifies these equations?

If $d_0(t)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} d_0(t) \in D_{d_0(t)}, \quad \forall t \in I \tag{15}$$

then, (8) is equivalent to the simpler condition

$$\xi(t) \in \mathfrak{g}^{d_0(t)}. \tag{16}$$

We shall call a gauge d_0 satisfying (15) a *non-holonomic gauge* and denote it as d_0^{NH}.

Following [2], given the base curve $\tilde{c}(t) \in Q/G$, a geometrically defined candidate for non-holonomic gauge d_0^{NH} fulfilling Eq. (15) is given by the *horizontal lift* of $\tilde{c}(t)$ with respect to the *non-holonomic connection*. The gauge d_0^{NH} obtained in this way is defined by

$$\begin{aligned} &\left(i^*_{d_0^{\mathrm{NH}}} I(d_0^{\mathrm{NH}}(t)) i_{d_0^{\mathrm{NH}}}\right)^{-1} \left(i^*_{d_0^{\mathrm{NH}}} J(\dot{d}_0^{\mathrm{NH}}(t))\right) = 0 \\ &\mathcal{A}^{\mathrm{Kin}}_{d_0^{\mathrm{NH}}}(\dot{d}_0^{\mathrm{NH}}(t)) = 0 \end{aligned} \tag{17}$$

where $i^*_{d_0^{\mathrm{NH}}} I(d_0^{\mathrm{NH}}(t)) i_{d_0^{\mathrm{NH}}} : \mathfrak{g}^{d_0^{\mathrm{NH}}} \longrightarrow \left(\mathfrak{g}^{d_0^{\mathrm{NH}}}\right)^*$ and $\mathcal{A}^{\mathrm{Kin}} : TQ \longrightarrow \mathfrak{U}$ denotes a $\mathfrak{U}$-valued 1-form that projects $\mathfrak{U}_q$ onto itself and has D_q as kernel. The subbundle $\mathfrak{U} \subset TQ$ can be defined to be, at each $q \in Q$, the (kinetic energy metric) orthogonal complement of $(\mathfrak{g}^q)_Q(q)$ within the subspace $T_q(Orb_G(q))$: $T_q(Orb_G(q)) = (\mathfrak{g}^q)_Q(q) \overset{\perp}{\oplus} \mathfrak{U}_q$ (see [2] for details).

In this case, the gauge factor $d_0^{\mathrm{NH}}(t)$ of the solution $c(t)$ can be *kinematically determined* from the base-controlled dynamics' $\tilde{c}(t)$. We also have

Proposition 2.13. *Let $d_0^{\mathrm{NH}}(t)$ be a* non-holonomic gauge *and consider the associated body momentum $\Pi(t)$ given by (5). The following holds:*

- the *constraints* read $g^{-1}\dot{g}(t) \in \mathfrak{g}^{d_0^{\mathrm{NH}}(t)}$,
- the *reconstruction* of $g(t)$ from $i^*_{d_0^{\mathrm{NH}}}\Pi(t)$ is:

$$g^{-1}\dot{g}(t) = \left(i^*_{d_0^{\mathrm{NH}}} \circ I(d_0^{\mathrm{NH}}(t)) \circ i_{d_0^{\mathrm{NH}}}\right)^{-1} \left(i^*_{d_0^{\mathrm{NH}}}\Pi(t) - i^*_{d_0^{\mathrm{NH}}} J(\dot{d}_0^{\mathrm{NH}})(t)\right)$$
$$\overset{\text{Eq. (17)}}{=} \left(i^*_{d_0^{\mathrm{NH}}} \circ I(d_0^{\mathrm{NH}}(t)) \circ i_{d_0^{\mathrm{NH}}}\right)^{-1} i^*_{d_0^{\mathrm{NH}}}\Pi(t),$$

- the equation of motion for $\Pi(t)$ reads

$$i^*_{d_0^{\mathrm{NH}}(t)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\Pi(t) + ad^*_{\left(i^*_{d_0^{\mathrm{NH}}}\circ I(d_0^{\mathrm{NH}}(t))\circ i_{d_0^{\mathrm{NH}}}\right)^{-1}\left(i^*_{d_0^{\mathrm{NH}}}\Pi(t)\right)}\Pi(t)\right) = 0,$$

- which is coupled to the constraint equation for $\Pi(t)$:

$$I_0^{-1}(t)(\Pi(t) - J(\dot{d}_0^{\mathrm{NH}})) \in \mathfrak{g}^{d_0^{\mathrm{NH}}(t)}.$$

Remark 2.14 (*No Constraints and the Mechanical Gauge*). Note that when $D = TQ$, i.e., when there are no constraints, the non-holonomic connection coincides with the *mechanical connection* (see for example [9] and the references therein) and, thus, the non-holonomic gauge reduces to the *mechanical gauge* d_0^{Mech} defined by

$$J(\dot{d}_0^{\mathrm{Mech}}(t)) = 0. \tag{18}$$

3. Special cases

3.1. The conserved momentum case

Here we describe base-controlled systems with no additional D-constraints, but whose motion is governed by *momentum conservation*. This case encodes an important class of systems in which the fiber motion is induced from the base in order to keep the momentum constant. In Section 4.2, we shall apply this description to the study of reconstruction phases for this systems.

There are two ways of encoding this *conserved momentum* case in the general D-constrained case described above. One is to think that $D = TQ$ and the momentum J as giving conserved quantities due to *horizontal symmetries* of the whole G (see [2] and Section 4.4). Another, is to think

$$J(\dot{c}) = \mu = \text{const.}$$

as an *affine constraint on the system* (see Section 3.2). Both strategies lead to the same results that we shall derive below in a (third possible) direct way, by analyzing the corresponding equations of motion.

Since no D-constraints are present in the system, we only need to assume $(H2)$ and $(DH2)$ from Sections 2.1 and 2.2, respectively. From these, using the variational techniques of 2.3, it follows that the momentum map J is conserved along the physical motion of the system $c(t) \in Q$, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t} J(\dot{c}(t)) = 0, \quad \forall t. \tag{19}$$

The *non-autonomous, second-order equations of motion* for $g(t)$, derived from (19), read

$$0 = ad^*_{\xi(t)}(I_0(t)\xi(t) + J_0(t)) + I_0(t)\frac{\mathrm{d}}{\mathrm{d}t}(\xi(t)) + \frac{\mathrm{d}}{\mathrm{d}t}(I_0(t))\xi(t) + \frac{\mathrm{d}}{\mathrm{d}t}J_0(t) \tag{20}$$

with $\xi(t) = g^{-1}\frac{\mathrm{d}}{\mathrm{d}t}g(t) \in \mathfrak{g}$ and initial values $(g(t_1), \dot{g}(t_1))$ fixed by (2). We shall now focus on the Hamiltonian structure of the equations of motion.

Let d_0 denote any gauge. Since I_q is a linear isomorphism for each $q \in Q$, the map sending $\xi \mapsto \Pi$ defined by Eq. (5), which can be seen as a *time-dependent Legendre transformation*, is invertible for all t:

$$\xi(t) = I_0^{-1}(t)(\Pi - J_0(t)). \tag{21}$$

We also see that Eq. (19) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\Pi(t) = -ad^*_{I_0^{-1}(t)(\Pi(t)-J_0(t))}\Pi(t). \tag{22}$$

We will now transform Eq. (20) into first-order non-autonomous equations on T^*G making use of underlying geometrical structures. Recall that T^*G is isomorphic as a vector bundle to $G \times \mathfrak{g}^*$ via *left translations*, i.e., by taking *body coordinates* [1,7]. Also recall the two maps $G \times \mathfrak{g}^* \overset{L}{\underset{\pi}{\rightrightarrows}} \mathfrak{g}^*$ described in Appendix B. We can now state the following

Proposition 3.1. *Let $g(t)$ be a curve in G and $\Pi(t) = I_0(t)g^{-1}\frac{\mathrm{d}}{\mathrm{d}t}g(t) + J_0(t)$. The curve $g(t)$ is a solution of (20) iff the curve $(g(t), \Pi(t))$ is an integral curve of the time-dependent vector field*

$$X(g, \Pi, t) = (g(I_0^{-1}(t)(\Pi - J_0(t))), -ad^*_{I_0^{-1}(t)(\Pi-J_0(t))}\Pi)$$

on $G \times \mathfrak{g}^*$ ($\sim T^*G$). In this case, if $L(g(t_1), \Pi(t_1)) = Ad^*_{g(t_1)}\Pi(t_1) = \mu$, then $(g(t), \Pi(t)) \in L^{-1}(\mu) \approx G$ for all $t \in I$.

Remark 3.2 (*Time-Dependent Reduction*). Recall that we started with an, a priori, $2 \times \dim G$-dimensional problem, defined by the non-autonomous second-order equation (20) for $g(t)$. Now, due to the conservation of the momentum J, we were also able to reduce the dimensionality to $\dim G = \dim(L^{-1}(\mu))$ because $\Pi(t)$ must be $Ad^*_{g^{-1}(t)}\mu$.

From the above proposition (equiv. form Eq. (22)) we have that

$$\Pi(t) \in O_\mu \subset \mathfrak{g}^*$$

with O_μ denoting the *G-coadjoint orbit* through μ in $\mathfrak{g}^*$. So, finally, to solve for $g(t) \in G$:

(1) we have to solve the *non-autonomous first-order differential equation* (22) on O_μ to obtain $\Pi(t)$ and
(2) then *reconstruct* $g(t)$ from $\Pi(t)$ in the G_μ-bundle $L^{-1}(\mu) \approx G \longrightarrow O_\mu$.

This last step is studied in Section 4.2 below (see also Appendix B).

Remark 3.3 (*Characterizing the Reduced Dynamics in O_μ*). In this paper, we elaborate on the factorization of the solution $c(t) = g(t)d_0(t)$ and we give *reconstruction* phase formulas for $g(t)$ associated to (horizontal) conservation laws. This last corresponds to step (2) above and is of geometric (kinematic) nature. Now, since these reconstruction formulas involve the *geometry* of the reduced dynamic's solution curve $\Pi(t) \in O_\mu$ (e.g. knowing that it closes after a certain time T and the geometry of this loop in O_μ), in order to fully characterize the solution $c(t)$ at times T when the formulas yield simple closed expressions, we have to deal with the problem of studying the solutions of the non-autonomous equation (22). This problem corresponds to step (1) above and is of *dynamical* nature. In Ref. [3], $G = SO(3)$ and a dynamical analysis of the reduced solution on the sphere $O_\mu = S^2_{\|\mu\|}$ was carried out by considering (non-conserved) energy level sets. In our general G setting, obtaining such a characterization is more involved. One strategy for studying $\Pi(t)$ consists in considering a *Hamiltonian* structure for the non-autonomous equation (22). This can be done, as usual [1], by adding time t and momentum conjugated to time E as variables. The resulting symplectic phase space is $O_\mu \times \mathbb{R} \times \mathbb{R}^* \ni (\Pi(t), t, E(t))$, where the new variable E becomes related by the corresponding Hamiltonian equations to the (non-conserved) kinetic energy K of the mechanical system (see Appendix A) via

$$E(t) = -K\left(\frac{\mathrm{d}}{\mathrm{d}t}c(t)\right) + \langle \Pi(t), I_0^{-1}(t)J_0(t)\rangle.$$

Consequently, by knowing the evolution of the energy K we can obtain (part of) the desired information about the dynamics of $\Pi(t)$.

3.2. *Affine D-constraints*

In this subsection we shall follow [2] and [4] to show how to handle *affine D-constrained controlled* systems. By an affine D-constraint we mean one of the type

$$\mathfrak{A}_q^D(\dot{q}) = \gamma(q, t) \tag{23}$$

where $\mathfrak{A}^D : TQ \longrightarrow TQ$ is a linear fiber projector defining an Eheresmann connection with $\mathrm{Ker}\,\mathfrak{A}^D = D \subset TQ$. We shall denote, as usual, the vertical subbundle by $V = \mathrm{Im}\,\mathfrak{A}^D \subset TQ$. The field $\gamma(q, t)$ is then vertical valued, that is, $\gamma(q, t) \in V_q \forall q, t$. Since our setting involves the geometry of the principal G-bundle $Q \longrightarrow Q/G$, we assume the following compatibility conditions to hold:

(i) $\mathfrak{A}^D$ is G-invariant, that is $\rho_{g*q} \circ \mathfrak{A}_q^D = \mathfrak{A}_{g\cdot q}^D \circ \rho_{g*q}$,
(ii) γ is G-invariant, that is $\gamma(g \cdot q, t) = \rho_{g*q}\gamma(q, t)$.

From the G-invariance of $\mathfrak{A}^D$ follows the G-invariance of D. We further assume the *dimension condition* on D, namely, $(H1)$ of Section 2.1. Now, we consider the *affine* version of the Lagrange–D'Alambert principle present in [4]:

PAff The curve $q(t)$ is a solution for the above stated non-holonomic affine-constrained system iff $\dot{q}(t)$ satisfies the affine constraints (23) and if for any variation $q(t, s)$ with fixed end points such that $\delta q \in D_q$, then

$$\delta \int_{t_1}^{t_2} L(q, \dot{q})\mathrm{d}t = 0.$$

As in Section 2.3, we adapt this variational formulation to the *base-controlled case* by considering only *vertical* variations $c(t, s) = g(t, s) \cdot d_0(t)$ for any gauge $d_0(t)$.

From this, it follows

Proposition 3.4. *The equations for $g(t)$ in order for $c(t) = g(t) \cdot d_0(t)$ to be a solution for the affine-constrained and controlled system satisfying* (i) *and* (ii) *described above are: the same* equations of motion (12) *for $\xi = g^{-1}\dot{g}$ as in the linear (non-affine) constraint case plus the* constraint equation

$$\mathfrak{A}_{d_0(t)}^D\left[\dot{d}_0(t) + (g^{-1}\dot{g})_Q(d_0(t))\right] = \gamma(d_0(t), t). \tag{24}$$

The fact that the equation of motion for $g^{-1}\dot{g}$ is the same for the affine and linear cases is already commented, in terms of the *non-holonomic momentum equation*, in [2] (see page 27).

As before, we can simplify the constraint equation by choosing suitable gauges d_0. In a *non-holonomic gauge* d_0^{NH}, Eq. (24) become

$$\mathfrak{A}_{d_0^{\mathrm{NH}}(t)}^D\left[(g^{-1}\dot{g})_Q(d_0^{\mathrm{NH}}(t))\right] = \gamma(d_0^{\mathrm{NH}}(t), t)$$

because $\mathfrak{A}_{d_0^{\mathrm{NH}}(t)}^D(\dot{d}_0^{\mathrm{NH}}(t)) = 0$. But, if we define an *affine non-holonomic gauge* d_0^{Aff} to be one satisfying

$$\mathfrak{A}_{d_0^{\mathrm{Aff}}(t)}^D(\dot{d}_0^{\mathrm{Aff}}(t)) = \gamma(d_0^{\mathrm{Aff}}(t), t) \tag{25}$$

then, Eq. (24) reads,

$$g^{-1}\dot{g} \in \mathfrak{g}^{d_0^{\mathrm{Aff}}(t)}$$

which is simpler to handle. Notice that Eq. (25) plus the requirement $\pi(d_0^{\mathrm{Aff}}(t)) = \tilde{c}(t)$ do not determine $d_0^{\mathrm{Aff}}(t)$ uniquely since $\dim D$ can be grater than $\dim B$. On the other hand, when the field $\gamma = 0$, a non-holonomic gauge is an affine gauge.

In Section 5.2, we apply this general considerations to study the motion of a *controlled* ball on a rotating turntable.

3.3. The case G abelian

We now illustrate the structure of the equations in the case G is abelian. This allows us to isolate the contribution to the motion coming from the non-trivial geometry of the vector bundle $\mathfrak{g}^D$ from the Lie algebraic part of the equations of motion (i.e. terms involving ad).

When G is abelian, Ad_g is the identity for all $g \in G$, and thus

- $\mathfrak{g}^{g\cdot q} = \mathfrak{g}^q\ \forall g \in G$, i.e., the subspaces $\mathfrak{g}^q$ are vertically constant in Q, thus $\mathfrak{g}^{d_0(t)} = \mathfrak{g}^{c(t)}$ and $i^*_{c(t)} = i^*_{d_0(t)}$,
- $I(g\cdot q) = I(q)$, thus $I(c(t)) = I(d_0(t))$,
- $J(\dot{c}) = I(d_0(t))g^{-1}\dot{g}(t) + J(\dot{d}_0) = \Pi(t)$,
- the equation of motion reads

$$i^*_{d_0(t)}\left(\frac{\mathrm{d}}{\mathrm{d}t}J(\dot{c})\right) = i^*_{d_0(t)}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(I(d_0(t))g^{-1}\dot{g}(t) + J(\dot{d_0})\right)\right] = 0, \tag{26}$$

- the constraint equation in a *non-holonomic gauge* stays as

$$g^{-1}\dot{g}(t) \in \mathfrak{g}^{d_0^{\mathrm{NH}}(t)} = \mathfrak{g}^{c(t)}.$$

By Eq. (17), the constraint equation in terms of $J(\dot{c})$ reads

$$I^{-1}(d_0^{\mathrm{NH}}(t))(J(\dot{c}) - J(\dot{d}_0^{\mathrm{NH}})) = \left(i^*_{d_0^{\mathrm{NH}}} \circ I(d_0^{\mathrm{NH}}) \circ i_{d_0^{\mathrm{NH}}}\right)^{-1}\left(i^*_{d_0^{\mathrm{NH}}} J(\dot{c})\right) \in \mathfrak{g}^{d_0^{\mathrm{NH}}(t)}. \tag{27}$$

Remark 3.5 (*Base of the $\mathfrak{g}^D$ Bundle*). Since $\mathfrak{g}^{g\cdot q} = \mathfrak{g}^q$ for abelian G, the vector bundle $\mathfrak{g}^D \longrightarrow Q$ descends to a vector bundle over the shape space $\mathfrak{g}^D \longrightarrow Q/G$. In this context, the objects $i^*_{c(t)} = i^*_{d_0(t)} = i^*_{\tilde{c}(t)}$ and $I(c(t)) = I(d_0(t)) = I(\tilde{c}(t))$ really depend on the base curve $\tilde{c}(t) \in Q/G$.

Now, we want to re-write the equation of motion for the momentum $J(\dot{c})$ in a usual first-order differential form. As in Section 2.4.2, consider a linear isomorphism $T_t : \mathfrak{g}^* \stackrel{\sim}{\longrightarrow} \mathfrak{g}^*$ taking the initial fiber $(\mathfrak{g}^{d_0^{\mathrm{NH}}(t_1)})^*$ to $(\mathfrak{g}^{d_0^{\mathrm{NH}}(t)})^*$,

$$i^*_{d_0^{\mathrm{NH}}(t)} \circ T_t = T_t \circ i^*_{d_0^{\mathrm{NH}}(t_1)}.$$

Eq. (26) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(i^*_{d_0^{\mathrm{NH}}(t)} J(\dot{c})\right) = [\dot{T}\, T^{-1}, i^*_{d_0^{\mathrm{NH}}(t)}]\left(J(\dot{c})\right), \tag{28}$$

which is equivalent to the corresponding expressions in terms of moving basis of [2]. The above equation states how the non-triviality of the bundle $\left(\mathfrak{g}^D\right)^*$ affects the evolution of the projected momentum $i^*_{d_0^{\mathrm{NH}}(t)} J(\dot{c})$. Note that even when the bundle $\mathfrak{g}^D$ is trivializable, but not directly trivial, the corresponding equation of motion also contains non-zero $\dot{T}\, T^{-1}$ term.

Remark 3.6 (*Trivial $\mathfrak{g}_I^D$ Bundle*). When $\mathfrak{g}_I^D$ it is directly *trivial*, the above equation reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(i^*_{d_0^{\mathrm{NH}}(t)} J(\dot{c})\right) = 0$$

so it gives a *conservation law* related to the given base curve $\tilde{c}(t)$.

More explicitly, let $\{e^i_{d_0^{\mathrm{NH}}(t)}\}_{i=1}^{\dim \mathfrak{g}^{d_0^{\mathrm{NH}}(t)}}$ be a (moving) basis for the fiber $\mathfrak{g}^{d_0^{\mathrm{NH}}(t)}$ along the gauge curve $d_0^{\mathrm{NH}}(t)$. Then, constraints (28) for $J(\dot{c})$ imply that

$$J(\dot{c}) = \sum_{i=1}^{\dim \mathfrak{g}^{d_0^{\mathrm{NH}}(t)}} \lambda_i(t)\, I(d_0^{\mathrm{NH}}(t)) e^i_{d_0^{\mathrm{NH}}(t)} + J(\dot{d}_0^{\mathrm{NH}}) \tag{29}$$

for some time-dependent coefficients $\lambda_i(t) \in \mathbb{R}$ to be determined. From (26), we have that the $\lambda_i(t)$'s must satisfy

$$A(t) \dot{\overrightarrow{\lambda}}(t) = -B(t) \overrightarrow{\lambda}(t) - \overrightarrow{c}(t)$$

where the time-dependent real $(\dim \mathfrak{g}^{d_0^{\mathrm{NH}}(t)} \times \dim \mathfrak{g}^{d_0^{\mathrm{NH}}(t)})$ matrices $A(t)$ and $B(t)$ are defined by

$$A_{ij}(t) = \left\langle I(d_0^{\mathrm{NH}}(t)) e^i_{d_0^{\mathrm{NH}}(t)}, e^j_{d_0^{\mathrm{NH}}(t)} \right\rangle =: I_{ij}^{\{e^k_{d_0^{\mathrm{NH}}(t)}\}}$$

$$B_{ij}(t) = \left\langle \frac{\mathrm{d}}{\mathrm{d}t} \left(I(d_0^{\mathrm{NH}}(t)) e^j_{d_0^{\mathrm{NH}}(t)} \right), e^i_{d_0^{\mathrm{NH}}(t)} \right\rangle$$

and the $\dim \mathfrak{g}^{d_0^{\mathrm{NH}}(t)}$ real vector $\overrightarrow{c}(t)$ by

$$c_j(t) = \left\langle \frac{\mathrm{d}}{\mathrm{d}t} J(\dot{d}_0^{\mathrm{NH}}), e^j_{d_0^{\mathrm{NH}}(t)} \right\rangle.$$

Note that A is symmetric and invertible. If we solved these equations for $J(\dot{c})(t)$, then the reconstruction of $g(t)$ from it is straightforward because, since G is abelian, we can make use of the exponential map $\exp : \mathfrak{g} \longrightarrow G$. This yields

$$\begin{aligned} g(t) &= \exp\left(\int_{t_1}^t \mathrm{d}s\, I(d_0^{\mathrm{NH}})^{-1} \left(J(\dot{c})(s) - J(\dot{d}_0^{\mathrm{NH}})(s) \right) \right) \\ &= \exp\left(\int_{t_1}^t \mathrm{d}s\, \left(i^*_{d_0^{\mathrm{NH}}} \circ I(d_0^{\mathrm{NH}}) \circ i_{d_0^{\mathrm{NH}}} \right)^{-1}_{(s)} \left(i^*_{d_0^{\mathrm{NH}}(s)} J(\dot{c})(s) \right) \right). \end{aligned} \tag{30}$$

Remark 3.7 (*Mechanical Connection Phase Formula*). As G is abelian, the above expression yields

$$c(t) = \exp\left(\int_{t_1}^t \mathrm{d}s\, I(d_0)^{-1}_{(s)} J(\dot{c})(s) \right) \cdot g_{\mathrm{Mech}}(t) \cdot d_0(t_1)$$

with

$$g_{\mathrm{Mech}}(t) = \exp\left(- \int_{t_1}^t \mathrm{d}s\, I(d_0^{\mathrm{NH}})^{-1} J(\dot{d}_0^{\mathrm{NH}})(s) \right)$$

such that $g_{\mathrm{Mech}}(t) \cdot d_0^{\mathrm{NH}}(t) = \mathrm{Hor}_{\mathrm{Mech}}(\tilde{c})(t)$ gives the horizontal lift of $\tilde{c}(t) \in B$ with respect to the *mechanical connection* (18) (see also Section 4.1). Notice that the equation of motion for $J(\dot{c})$ (but not the constraint equation[6]) is the same in any gauge $d_0(t)$.

Finally, to better understand how the geometry of the bundle $\mathfrak{g}^D$ enters the equations of motion for $J(\dot{c})$, we restrict ourselves to the interesting case in which the horizontal space with respect to the *non-holonomic connection* is (kinetic energy metric) orthogonal to the whole vertical subspace $T\,Orb_G$ within TQ. In this case, a mechanical gauge $d_0(t)$,

6 For a general non *non-holonomic* gauge, constraint equation becomes the *gauge covariant* equation (8).

for which $J(\dot{d}_0) = 0$, is also a non-holonomic one and Eq. (26) yields the *parallel transport equation*:

$$D_{\dot{d}_0} \overrightarrow{p} \equiv \frac{\mathrm{d}}{\mathrm{d}t} p^i - \sum_{j=1}^{\dim \mathfrak{g}^{d_0(t)}} \gamma^i_j \, p^j = 0, \quad \forall\, 1 \leq i \leq \dim \mathfrak{g}^{d_0(t)} \tag{31}$$

for

$$p^i(t) := \left\langle J(\dot{c}), e^i_{d_0(t)} \right\rangle = \sum_{j=1}^{\dim \mathfrak{g}^{d_0(t)}} \lambda_j(t) \left\langle I(d_0(t)) e^j_{d_0(t)}, e^i_{d_0(t)} \right\rangle$$

being the coordinates of $J(\dot{c})$ in a basis of $\mathfrak{g}^*$ dual to a basis $\{e^k_{d_0(t)}\}_1^{\dim \mathfrak{g}}$ for which

$$\left\langle I(d_0(t)) e^i_{d_0(t)}, e^{\acute{i}}_{d_0(t)} \right\rangle = 0 \quad \forall\, 1 \leq i \leq \dim \mathfrak{g}^{d_0(t)}, \dim \mathfrak{g}^{d_0(t)} + 1 \leq \acute{i} \leq \dim \mathfrak{g}. \tag{32}$$

Note that, above, for $\dim \mathfrak{g}^{d_0(t)} + 1 \leq \acute{i} \leq \dim \mathfrak{g}$ then $p^{\acute{i}} = 0$ by the orthogonality condition (32) and because (iff) the constraints (29) are fulfilled. The *linear connection coefficients* γ^i_k are defined by

$$D_{(\dot{d}_0)} e^i_{d_0(t)} := \frac{\mathrm{d}}{\mathrm{d}t} e^i_{d_0(t)} = \sum_{k=1}^{\dim \mathfrak{g}} \gamma^i_k \, e^k_{d_0(t)}.$$

Consequently, for this case, the time evolution of $J(\dot{c})$ is *geometrically determined*: it moves *parallel-transported* along the base curve $\tilde{c}(t) \in Q/G$ in the bundle $\mathfrak{g}^D \longrightarrow Q/G$ of Remark 3.5 (see also [2]). On the other hand, as noticed in Remark 3.6, when the involved geometry is trivial, i.e. $\mathfrak{g}^D = Q \times V$ with constant $V \subset \mathfrak{g}$, then $i^*_V J(\dot{c})$ is a *conserved quantity*. Indeed, since $\mathfrak{g}$ is abelian, such a V defines a subalgebra and we are in the case described in Section 4.4.

In Section 5.1, we apply these general considerations to study the motion of a *base-controlled* vertical rotating disk.

3.4. The trivial bundle case $Q = G \times B$

To illustrate on how the controlled base variables induce motion on the group variables, we now focus on the case in which $Q = G \times B$, i.e., $Q \longrightarrow Q/G$ is a trivial principal G- bundle. Recall that we are considering the natural *left* G-action on $G \times B$. In this case,

$$TQ = TG \oplus TB$$

and thus, by hypothesis $(H1)$ of Section 2.1,

$$D_{(b,g)} = T_b B \oplus S_{(b,g)}$$

with $S_{(b,g)} := T_g G \cap D_{(b,g)}$ as usual. Note that, since D is G-invariant, for each $b \in B$, $S_{(b,g)}$ defines a G-invariant distribution on G which, in turn, is fixed by the subspace $S_{(b,e)} \subset T_e G = \mathfrak{g}$. So D is characterized by a smooth map $B \longrightarrow \mathrm{Gr}_{\dim S}(\mathfrak{g}) :=$ {Grassmanian of $\dim S$ subspaces of $\mathfrak{g}$} or, equivalently, by a vector bundle

$$V = \bigcup_{b \in B} S_{(b,e)} \longrightarrow B. \tag{33}$$

Conversely, if $V \longrightarrow B$ is a vector bundle over the base B with fibers $V_b \subset \mathfrak{g}$, it defines a G-invariant distribution D on $G \times B$ by setting $S_{(b,g)} = L_{g*e} V_b$. The vector bundle $S \subset D$ thus corresponds to the map

$$\begin{aligned} G \times B &\longrightarrow \mathrm{Gr}_{\dim S}(\mathfrak{g}) \\ (b, g) &\longmapsto L_{g*e} V_b. \end{aligned}$$

Now, the subspace (recall Proposition 2.7) $\mathfrak{g}^{(b,g)}$ is given by

$$\mathfrak{g}^{(b,g)} = \{X \in \mathfrak{g}, \exists Y \in S_{(b,e)}; X = Ad_g Y\}$$

so,

$$\mathfrak{g}^{(b,g)} = Ad_g\ \mathfrak{g}^{(b,e)}$$
$$\mathfrak{g}^{(b,e)} = S_{(b,e)}.$$

At this point, we make an assumption on the metric on $Q = G \times B$:

(HM) Suppose that we have a smooth map

$$B \longrightarrow \{\text{Left invariant metrics on } G\} \simeq \{\text{metrics on } \mathfrak{g}\}$$
$$b \mapsto (,)_b\,.$$

The metric $k^Q(,)$ on Q is assumed to be given by

$$k^Q_{(b,g)}\left(\left(\dot{b}_1, \dot{g}_1\right), \left(\dot{b}_2, \dot{g}_2\right)\right) = k^B_b(\dot{b}_1, \dot{b}_2) + (g_1^{-1}\dot{g}_1, g_2^{-1}\dot{g}_2)_b$$

for $k^B(,)$ being a metric on B.

Remark 3.8 (*Applicability*). This kind of metric on $Q = G \times B$ is the one present on typical examples (see [2]). See also the examples of Section 5.

Assuming (HM), the momentum map $J : TQ \longrightarrow \mathfrak{g}^*$ corresponding to the left G-action on Q is

$$J(\dot{b}, \dot{g}) = Ad^*_g\, \Psi_b(g^{-1}\dot{g})$$

with $\Psi_b : \mathfrak{g} \longrightarrow \mathfrak{g}^*$ denoting the isomorphism defined by the metric $(,)_b$ on $\mathfrak{g}$. The inertia tensor $I_{(b,g)} : \mathfrak{g} \longrightarrow \mathfrak{g}^*$ takes the form

$$I_{(b,g)} = Ad^*_g \circ \Psi_b \circ Ad_{g^{-1}}.$$

Note that we have a natural lift $d_0^{\mathrm{NH}}(t) = (\tilde{c}(t), e) \in B \times G$ for a curve $\tilde{c}(t) \in B$. This gauge $d_0^{\mathrm{NH}}(t)$ defines a non-holonomic gauge as defined in Section 2.4.3. In fact, this $d_0^{\mathrm{NH}}(t)$ coincides with the horizontal lift of $\tilde{c}(t)$ from $(\tilde{c}(t_1), e)$ with respect to the non-holonomic connection of [2]. Moreover, it is also a mechanical gauge (18).

For this gauge choice, the inclusion

$$i_{d_0^{\mathrm{NH}}(t)} : \mathfrak{g}^{d_0^{\mathrm{NH}}(t)} = S_{(\tilde{c}(t),e)} \hookrightarrow \mathfrak{g}$$

depends only on the base curve $\tilde{c}(t) \in B$ and coincides with the inclusion

$$i_{\tilde{c}(t)} : V_{\tilde{c}(t)} \hookrightarrow \mathfrak{g}$$

where $V_{\tilde{c}(t)} = S_{(\tilde{c}(t),e)}$ is the fiber of the vector bundle (33). The curve $c(t)$ describing the motion on the constrained and controlled system on Q will thus be

$$c(t) = (\tilde{c}(t), g(t)) = g(t) \cdot d_0^{\mathrm{NH}}(t)$$

and

$$J(\dot{c}) = Ad^*_g\, I_{(\tilde{c},e)}(g^{-1}\dot{g}) = Ad^*_g\, \Psi_{\tilde{c}(t)}(g^{-1}\dot{g}).$$

In this case, equations of motion (11) read

$$i^*_{c(t)}\left(\frac{\mathrm{d}}{\mathrm{d}t} J(\dot{c})\right) = 0$$

or, equivalently,

$$i^*_{\tilde{c}(t)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\Psi_{\tilde{c}(t)}(g^{-1}\dot{g})\right) + ad^*_{g^{-1}\dot{g}}\, \Psi_{\tilde{c}(t)}(g^{-1}\dot{g})\right) = 0. \tag{34}$$

The constraints for $g(t)$ are

$$g^{-1}\dot{g}(t) \in \mathfrak{g}^{(\tilde{c}(t),e)} = S_{(\tilde{c}(t),e)} = V_{\tilde{c}(t)}. \tag{35}$$

Eq. (34) can be re-written using a moving basis system on the vector bundle $V \longrightarrow B$ as done in the previous section, yielding the local expression of the non-holonomic momentum equations of [2] evaluated along $\tilde{c}(t)$.

Let us simplify the situation a bit more to try to isolate the Lie-algebraic (vertical) contribution to the system's motion from the $\mathfrak{g}^D$-geometric (horizontal) contribution studied in the previous section. In case the bundle $V \longrightarrow B$ is *trivial*, that is $S_{(b,e)} = S_0 \subset \mathfrak{g}$ for all $b \in B$, then

$$i^*_{\tilde{c}(t)} = i^*_0 \; \forall t$$

and so Eq. (34) reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(i^*_0 \Psi_{\tilde{c}(t)}(g^{-1}\dot{g})\right) = -i^*_0\left(ad^*_{g^{-1}\dot{g}}\Psi_{\tilde{c}(t)}(g^{-1}\dot{g})\right)$$

which is an equation for $\Psi_{\tilde{c}(t)}(g^{-1}\dot{g})$, coupled to the constraint equation (35) for $g^{-1}\dot{g}$. Its algebraic structure is still hard to handle in general. If we wanted to solve the above (general) equation by using usual Lie-algebraic properties of $\mathfrak{g}$, then we would need to assume some additional condition on how the subspace S_0 changes when moving vertically along the fiber $(\tilde{c}(t), e) \rightsquigarrow (\tilde{c}(t), g)$.

Suppose, then, that S_0 is Ad_G invariant. It follows that $\mathfrak{g}^{c(t)} = \mathfrak{g}^{\tilde{c}(t)} = S_0$ and that $S_0 \subset \mathfrak{g}$ is a Lie subalgebra. By the constraints $g^{-1}\dot{g} \in S_0$ and the equation of motion (11) becomes the *conservation law* (as in Remark 3.6)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(i^*_0 J(\dot{c})\right) = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(i^*_0 \Psi_{\tilde{c}(t)}(g^{-1}\dot{g})\right) = -\left(ad^*_{g^{-1}\dot{g}} i^*_0 \Psi_{\tilde{c}(t)}(g^{-1}\dot{g})\right).$$

Although looking integrable, this equation is still hard to solve explicitly in general (see [7] for the rigid body $\mathfrak{g} = \mathfrak{so}(3)$ case). Nevertheless, in this situation, the dynamical factor $g(t)$ of $c(t)$ can be *reconstructed* from a solution of the above equation in S_0 yielding corresponding *phase formulas*, as described in Section 4 and Appendix B.

From the analysis of this section, we see that even under very favorable hypothesis on the geometry of Q and D, the equations of motion can be very complicated and we cannot continue with the general study of $c(t)$ (recall Remark 3.3). Nevertheless, if we require deeper compatibilities (as above) between D and the G-action, e.g. horizontal symmetries, in Sections 4.4 and 4.5 we shall show that further *phase formulas* can be given for characterizing the solution $c(t)$.

4. Reconstruction and phases

In the following, we focus on reconstruction phases [6] for both the full solution $c(t)$ and vertical (gauge-dependent) unknown $g(t)$. The interested reader can find various types of reconstruction phases in [8].

4.1. Gauges and phases in $Q \longrightarrow Q/G$ for D-constrained systems

Suppose that the base curve $\tilde{c}(t) \in Q/G$ is closed, $\tilde{c}(t_1) = \tilde{c}(t_2)$. Choice (17) for the *non-holonomic gauge* $d^{\mathrm{NH}}_0(t)$ provides us with a geometric phase in the motion of the system in Q as follows. Being defined as a horizontal lift, $d^{\mathrm{NH}}_0(t_2)$ coincides with the *holonomy* of the associated to the base curve $\tilde{c}(t)$ measured from the initial condition $d^{\mathrm{NH}}_0(t_1) = c(t_1)$ and with respect to the *non-holonomic connection*. Thus, the corresponding phase formula is

$$c(t_2) = g_{\mathrm{Dyn}}(t_2) \cdot g_{NH} \cdot d^{\mathrm{NH}}_0(t_1)$$

with g_{NH} uniquely defined by $d^{\mathrm{NH}}_0(t_2) = g_{NH} \cdot d^{\mathrm{NH}}_0(t_1)$ and where $g_{\mathrm{Dyn}}(t)$ is the solution of Eqs. (12) and (16), with $\xi(t) = g^{-1}_{\mathrm{Dyn}}\dot{g}_{\mathrm{Dyn}}$ and time-dependent coefficients evaluated along this gauge $d^{\mathrm{NH}}_0(t)$.

Another geometric phase g_{MP} appears when using the *mechanical gauge*. Let the gauge $d_0^{\mathrm{NH}}(t)$ be as above and $g_{\mathrm{Mech}}(t)$ be defined by requiring $\tilde{d}_0(t) := g_{\mathrm{Mech}}(t) \cdot d_0^{\mathrm{NH}}(t)$ to be the horizontal lift with respect to the *mechanical connection* (18) (see [9]) on Q with $g_{\mathrm{Mech}}(t_1) = e$. If we write $g_{\mathrm{Dyn}}(t) = g_{\tilde{D}}(t) \cdot g_{\mathrm{Mech}}(t)$, the corresponding equations of motion for the remaining dynamic contribution $g_{\tilde{D}}(t)$ are

$$i^*_{\tilde{d}_0(t)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(I(\tilde{d}_0(t))g_{\tilde{D}}^{-1}\dot{g}_{\tilde{D}}\right) + ad^*_{g_{\tilde{D}}^{-1}\dot{g}_{\tilde{D}}} I(\tilde{d}_0(t))g_{\tilde{D}}^{-1}\dot{g}_{\tilde{D}}\right) = 0 \tag{36}$$

which are simpler from the original ones (12) because the $J(\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{d}_0))$ term vanishes by (18). But the constraint equations (16) in terms of $g_{\tilde{D}}$ read

$$g_{\tilde{D}}^{-1}\dot{g}_{\tilde{D}} + \dot{g}_{\mathrm{Mech}} g_{\mathrm{Mech}}^{-1} \in \mathfrak{g}^{\tilde{d}_0(t)} \tag{37}$$

which are more complicated than the original ones for g_{Dyn}.

The relation between the above different gauge phases read

$$\begin{aligned} c(t_2) &= g_{\mathrm{Dyn}}(t_2) \cdot g_{NH} \cdot c(t_1) \\ &= g_{\tilde{D}}(t_2) \cdot g_{\mathrm{Mech}}(t_2) \cdot g_{NH} \cdot c(t_1) \\ &= g_{\tilde{D}}(t_2) \cdot g_{MP} \cdot c(t_1). \end{aligned}$$

with the second *geometric phase* being $g_{MP} = g_{\mathrm{Mech}}(t_2) \cdot g_{NH}$.

Remark 4.1 (*Simplifications from Different Gauges*). In the non-holonomic gauge, the constraint equations are simpler and, in turn, in the mechanical gauge the equations of motion become simpler. One would like to have both simplifications to hold, but this cannot be achieved in general since the horizontal lift with respect to the mechanical connection is not horizontal with respect to the non-holonomic connection for general D. Finally, we would like to observe that, in some situations, we have additional information about the D-constraints and the non-holonomic gauge becomes preferable (see, for example, the next sections).

4.2. Reconstruction phases for systems with conserved momentum

Now, we shall elaborate on the reconstruction of $g(t)$ for a solution $\Pi(t)$ in $O_\mu \subset \mathfrak{g}^*$, as described in Section 3.1 in case there are no D-constraints. A concrete example of the *phase formulas* we obtain below can be found in [3] for the motion of a self-deforming body.

Suppose that we have a solution $\Pi(t) = Ad^*_{g^{-1}(t)} J(\dot{c}) \in O_\mu$ for Eq. (22) with $\mu = J(\dot{c}) = \mathrm{const} \neq 0$ and that we chose a linear projector $P : \mathfrak{g} \twoheadrightarrow \mathfrak{g}_\mu$ satisfying

$$Ad_h \circ P = P \circ Ad_h. \tag{38}$$

From Appendix B, we know that we can then write

$$g(t) = h_D(t) \cdot g_G(t)$$

with the *geometric phase* g_G being the *horizontal lift* of $\Pi(t)$ with respect to connection defined by P in the G_μ-bundle $G \longrightarrow O_\mu$ and the *dynamic phase* $h_D \in G_\mu$ defined by

$$\frac{\mathrm{d}}{\mathrm{d}t} h_D h_D^{-1}(t) = P\left(I_{c(t)}^{-1}(J(\dot{c}) - Ad_g^* J_0(t))\right) \tag{39}$$

with $h_D(t_1) = e$. The last step follows from Eq. (21) for $g(t)$ where $I_{c(t)}$ denotes the inertia tensor evaluated along the physical motion $c(t)$.

Suppose now that $\mathfrak{g}$ has an Ad-invariant scalar product $(,)$, as considered in Appendix B. Let $u_1 = \frac{\Psi(\mu)}{\|\Psi(\mu)\|}$ and $\{u_i\}_{i=1}^{\dim \mathfrak{g}_\mu}$ denote an orthonormal basis with respect to $(,)$ of the vector subspace $\mathfrak{g}_\mu \subset \mathfrak{g}$. In this case, Eq. (39)

becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}h_D h_D^{-1}(t) = \frac{1}{\parallel \Psi(\mu)\parallel^2}(2K(\dot{c}(t)) - 2K_{\mathrm{int}}(t) + \langle J_0(t), I_0^{-1}(t)J_0(t)\rangle - \langle J_0(t), I_0^{-1}(t)\Pi(t)\rangle)\,\Psi(\mu) + \sum_{i=2}^{\dim \mathfrak{g}_\mu}\left[(u_i, I_{c(t)}^{-1}\mu) - (u_i, I_{c(t)}^{-1}Ad_g^* J_0(t))\right] u_i \tag{40}$$

where K represents the *kinetic energy* of the controlled system in Q (see Appendix A). When $d_0(t) = d_0^{\mathrm{Mec}}(t)$ is the *mechanical gauge* (18),

$$\frac{\mathrm{d}}{\mathrm{d}t}h_D h_D^{-1}(t) = \frac{1}{\parallel \Psi(\mu)\parallel^2}(2K(\dot{c}(t)) - 2K_{\mathrm{int}}(t))\,\Psi(\mu) + \sum_{i=2}^{\dim \mathfrak{g}_\mu}(u_i, I_{c(t)}^{-1}\mu)u_i. \tag{41}$$

Remark 4.2 (*Locked Inertia Tensor and Physical Information in h_D*). The above reconstruction phase formula, in the mechanical gauge, relates the dynamical phase h_D to the data of the locked inertia tensor $I_{c(t)}$ and the kinetic energy K, both along the physical solution curve $c(t)$ in Q, and to the gauge kinetic energy $K_{\mathrm{int}}(\dot{d}_0^{\mathrm{Mec}})$.

Remark 4.3 (*The Case $J(\dot{c}) = 0$*). In this case, the system's motion $c(t)$ coincides with the *mechanical gauge* $d_0^{\mathrm{Mech}}(t)$ motion because of (18). We thus say that the induced motion $c(t)$ is *geometrical* with respect to the base one $\tilde{c}(t)$ (see also Example 4.6).

Remark 4.4 (*The Case G Abelian*). In this case, $I_{c(t)} = I_{d_0^{\mathrm{Mec}}(t)}$ and, thus, the only dynamical (i.e. non-kinematical) information needed to evaluate formula (41) is the system's kinetic energy evolution $K(\dot{c}(t))$. In this case, $h_D(t)$ can also be easily integrated by means of the corresponding exponential map $\exp : \mathfrak{g}_\mu \longrightarrow G_\mu$.

4.3. Phases for D-constrained, purely kinematical systems

We recall from [2],

Definition. A constrained system (Q, L, G, D) is said to have *purely kinematical* (PK) constraints if $TQ = \mathrm{Ver} \oplus D$.

Since D is G-invariant, it defines a principal connection on $Q \longrightarrow Q/G$. Let A^D denote the corresponding $\mathfrak{g}$-valued 1-form on Q. The constraint equation for $c(t)$ then reads

$$A^D(\dot{c}) = 0$$

and the vertical equations of motion (12) are trivial since $\mathfrak{g}^q = 0$ for all q. So we have that

Proposition 4.5. *The motion for a base-controlled system $(Q, L, G, D, \tilde{c})$ for which D defines purely kinematical PK constraints, is of geometric nature with respect to $\tilde{c}$. In other words, the solution $c(t)$ is given by the horizontal lift of the base-controlled curve $\tilde{c}(t)$ with respect to the principal connection on $Q \longrightarrow Q/G$ defined by the constraint distribution D.*

Corollary. *If $\tilde{c}$ is closed in $[t_1, t_2]$, we then have a* geometric phase g_G *in the system's dynamics associated to the initial value $c(t_1)$ and defined by $g_G = \mathrm{Hol}(\tilde{c})$:*

$$c(t_2) = g_G \cdot c(t_1).$$

Example 4.6 (*Deforming Bodies with Zero Angular Momentum*). If we regard $J(\dot{c}) = 0$ as a D-constraint for the motion of a self deforming body, with J being the angular momentum map, then D coincides with the mechanical connection's horizontal space. From the above proposition, we recover the known fact [11] that global reorientation $g(t) \in SO(3)$ of such a body is *geometrical* with respect to the deformation $\tilde{c}(t)$.

4.4. Phases for D-constrained systems with horizontal symmetries

We now analyze a geometric-kinematical favorable case leading to phase formulas for the dynamical factor $g(t)$ of $c(t)$.

Definition *([2])*. A constrained system (Q, L, G, D) is said to have (full) *horizontal symmetries* (HS) if there exists a subgroup $H \subset G$ such that

(1) $\xi_Q(q) \in D_q \forall q \in Q$ when $\xi \in \mathfrak{h} := \mathrm{Lie}(H) \subset \mathfrak{g}$,
(2) (*Full* condition) $S_q := D_q \cap T_q\,(Orb_G(q)) = T_q\,(Orb_H(q))\ \forall q \in Q$.

Condition (2) above states that horizontal symmetries exhaust the whole vertical kinematics. The analysis we give below can be extended to the non-full case, i.e. by assuming only (1), but we keep hypothesis (2) for simplicity. Example 5.3 below illustrates the non-full case.

For an HS system, the bundle $\mathfrak{g}^D$ is the trivial one $Q \times \mathfrak{h}$. Since the inclusion map $i_q = i_{\mathfrak{h}} : \mathfrak{h} = \mathfrak{g}^q \hookrightarrow \mathfrak{g}$ becomes independent of the point q, Eq. (11) reads

$$i_{\mathfrak{h}}^* \left(\frac{\mathrm{d}}{\mathrm{d}t} (J(\dot{c})) \right) = \frac{\mathrm{d}}{\mathrm{d}t} (i_{\mathfrak{h}}^* J(\dot{c})) = 0. \tag{42}$$

Consequently, $i_{\mathfrak{h}}^* J(\dot{c})$ gives a *conserved quantity* during the motion of the system as at the end of Section 3.4. This projection $i_{\mathfrak{h}}^* J(\dot{c})$ can be interpreted as the part of the total momentum map which is compatible with the constraints (see also [2]).

Next, we shall enunciate a few results which follow from the definition of a system with full HS.

Proposition 4.7. *The following holds:*

- $H \subset G$ *is a* normal *subgroup of* G, *thus* $\mathfrak{h}$ *is* G*-invariant* $Ad_g \mathfrak{h} = \mathfrak{h}$,
- $i_{\mathfrak{h}}\, Ad_g = Ad_g\, i_{\mathfrak{h}}$,
- *For each* $q \in Q$, *let* $I_q^{\mathfrak{h}} = i_{\mathfrak{h}}^* \circ I_q \circ i_{\mathfrak{h}} : \mathfrak{h} \longrightarrow \mathfrak{h}^*$ *be the* restricted inertia tensor, *then*

$$I_{g\cdot q}^{\mathfrak{h}} = Ad_g^* I_q^{\mathfrak{h}} Ad_{g^{-1}}, \quad \forall g \in H.$$

We shall now describe the appearance of phase formulas for the dynamical factor $g(t)$ of the motion $c(t)$ of an HS system. First, recall that in a *non-holonomic gauge* $d_0^{\mathrm{NH}}(t)$ the constraint equation for the body velocity $\xi = g^{-1}\dot{g}$ becomes Eq. (16) which, for an HS system, reduces to

$$g^{-1}\dot{g}(t) \ \in \mathfrak{h} \tag{43}$$

for all t. From the other side, if we consider the non-holonomic *body momentum* $\Pi(t) \in \mathfrak{g}^*$ of Eq. (5), because of the constraint (43), we have that

$$g^{-1}\dot{g}(t) = \xi(t) = (I_0^{\mathfrak{h}})_{(t)}^{-1} \left(i_{\mathfrak{h}}^* \Pi(t) - i_{\mathfrak{h}}^* J(\dot{d}_0^{\mathrm{NH}}(t)) \right). \tag{44}$$

Thus, Eq. (12), equiv. Eq. (42), become

$$\frac{\mathrm{d}}{\mathrm{d}t}(i_{\mathfrak{h}}^* \Pi(t)) = -ad^*_{g^{-1}\dot{g}(t)}(i_{\mathfrak{h}}^* \Pi(t)) = -ad^*_{(I_0^{\mathfrak{h}})_{(t)}^{-1}(i_{\mathfrak{h}}^* \Pi(t) - i_{\mathfrak{h}}^* J(\dot{d}_0^{\mathrm{NH}}(t)))} i_{\mathfrak{h}}^* \Pi(t). \tag{45}$$

The above expressions are equivalent to

$$i_{\mathfrak{h}}^* \Pi(t) = Ad^*_{g^{-1}}(i_{\mathfrak{h}}^* J(\dot{c})) \tag{46}$$

because

$$i_{\mathfrak{h}}^* J(\dot{c}) = i_{\mathfrak{h}}^* Ad_g^*(\Pi(t)) = Ad_g^*(i_{\mathfrak{h}}^* \Pi(t)),$$

by Proposition 4.7. The constraint equation (43) can be also put in terms of $\Pi(t)$ as follows

$$I_0^{-1}(t)(\Pi(t) - J(\dot{d}_0^{\mathrm{NH}}(t))) = (I_0^{\mathfrak{h}})^{-1}_{(t)}\left(i_{\mathfrak{h}}^*\Pi(t) - i_{\mathfrak{h}}^*J(\dot{d}_0^{\mathrm{NH}}(t))\right) \in \mathfrak{h}. \tag{47}$$

Eqs. (45) and (47), both determine the dynamics of $\Pi(t) \in \mathfrak{g}^*$ from the initial value $\Pi(t_1) = J(\dot{c}) = \mu$.

Now, from (43) and $g(t_1) = e$ it follows that $g(t) \in H$ for all $t \in [t_1, t_2]$. Thus, from Eq. (46), we can deduce that

$$i_{\mathfrak{h}}^*\Pi(t) \in O^H_{i_{\mathfrak{h}}^*J(\dot{c})}$$

where $O^H_{i_{\mathfrak{h}}^*J(\dot{c})}$ denotes the H-*coadjoint orbit* in $\mathfrak{h}^*$ through the constant element $i_{\mathfrak{h}}^*J(\dot{c})$. We are then in the situation described in Appendix B (but with group H instead of G) and we can thus apply the usual *reconstruction procedure* of [6] for the group unknown $g(t) \in H$ from a solution $i_{\mathfrak{h}}^*\Pi(t) \in O^H_{i_{\mathfrak{h}}^*J(\dot{c})}$.

Remark 4.8 (*Initial Conditions*). When the initial conditions are $g(0) \neq e$ in G, so $c(0) = g(0) \cdot d_0^{\mathrm{NH}}(0)$, then Eq. (43) implies that $g(t) = g(0) \cdot g_H(t)$ where $g_H(t) \in H$ is the solution corresponding to the initial condition $g_H(0) = e$. Thus, below we shall focus on the $g(0) = e$ case.

Let $P : \mathfrak{h} \longrightarrow \mathfrak{h}_{i_{\mathfrak{h}}^*J(\dot{c})} = \mathrm{Lie}(H_{i_{\mathfrak{h}}^*J(\dot{c})})$ be a linear projector s.t. $P \circ Ad_g = Ad_g \circ P$ for all $g \in H$. Following Appendix B,

Proposition 4.9. *Keeping the notations introduced above, let $\Pi(t) \in \mathfrak{g}^*$ be a solution*[7] *of Eqs. 45, 47 and $i_{\mathfrak{h}}^*\Pi(t)$ its projection onto $\mathfrak{h}^*$. Then, the corresponding solution $g(t)$ of the reconstruction equation (44) which satisfies the constraints (43) with $g(0) = e$ is such that $g(t) \in H$ $\forall t \in I$ and*

$$g(t) = h_D(t)\, g_G(t).$$

Above, the geometric phase $g_G(t)$ *is the* horizontal lift *of $i_{\mathfrak{h}}^*\Pi(t) \in O^H_{i_{\mathfrak{h}}^*J(\dot{c})}$ from $g_G(0) = e$ with respect to the P-induced principal connection A_P on the principal $H_{i_{\mathfrak{h}}^*J(\dot{c})}$-bundle $H \xrightarrow{\pi} O^H_{i_{\mathfrak{h}}^*J(\dot{c})}$ and the* dynamic phase $h_D(t) \in H_{i_{\mathfrak{h}}^*J(\dot{c})}$ *is defined by the equation*

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}h_D h_D^{-1}(t) &= A_P\left(\frac{\mathrm{d}}{\mathrm{d}t}g\right)_g = P\left(Ad_{g(t)}(I_0^{\mathfrak{h}})^{-1}_{(t)}(i_{\mathfrak{h}}^*\Pi(t) - i_{\mathfrak{h}}^*J(\dot{d}_0^{\mathrm{NH}}(t)))\right) \\
&= P\left((I^{\mathfrak{h}}_{c(t)})^{-1}\left(i_{\mathfrak{h}}^*J(\dot{c}) - Ad^*_{g(t)}i_{\mathfrak{h}}^*J(\dot{d}_0^{\mathrm{NH}}(t))\right)\right)
\end{aligned} \tag{48}$$
$$h_D(0) = e.$$

Remark 4.10 (*Physical Content of h_D*). The above dynamical phase h_D depends on the (restricted) inertia tensor $I^{\mathfrak{h}}_{c(t)}$ and on the gauge internal momentum $Ad^*_{g(t)}i_{\mathfrak{h}}^*J(\dot{d}_0^{\mathrm{NH}}(t))$, both as seen from the reference system which is moving along the physical evolution $c(t) \in Q$. Moreover, if the non-holonomic gauge choice is the horizontal one (17), then

$$\frac{\mathrm{d}}{\mathrm{d}t}h_D h_D^{-1}(t) = P\left((I^{\mathfrak{h}}_{c(t)})^{-1}i_{\mathfrak{h}}^*J(\dot{c})\right)$$

only depends on $I^{\mathfrak{h}}_{c(t)}$.

[7] The problem of characterizing the geometry of the reduced solutions $i_{\mathfrak{h}}^*\Pi(t) \in O^H_{i_{\mathfrak{h}}^*J(\dot{c})}$ of Eq. (45) corresponds to the dynamical one described in Remark 3.3.

Remark 4.11 (*The Case* $i^*_{\mathfrak{h}}J(\dot{c}) = 0$). In this case, $g(t)$ coincides with the dynamical phase and is given by

$$g^{-1}\dot{g}(t) = -(I^{\mathfrak{h}}_{d_0^{\mathrm{NH}}(t)})^{-1} i^*_{\mathfrak{h}} J(\dot{d}_0^{\mathrm{NH}}(t))$$

since $i^*_{\mathfrak{h}}\Pi(t) = 0$ by (46). Nevertheless, the full motion $c(t)$ is *geometric* with respect to the base one $\tilde{c}(t)$. The reason is that $c(t)$ coincides with the horizontal lift d_0^{NH} of $\tilde{c}$ with respect to the *non-holonomic connection* [2] because of equation (17). Notice that this is true for *full* horizontal symmetries, i.e., when the conservation of $i^*_{\mathfrak{h}}J = 0$ exhausts the whole vertical equations of motion. This result generalizes the one of [11] (see Example 4.6) on the geometric nature of base-induced motion for zero momentum systems to the context of D-constrained systems with full horizontal symmetries.

When $\mathfrak{h}$ admits an Ad-invariant inner product, the dynamic phase equation can be also related to other mechanical magnitudes.

Proposition 4.12. *Keeping the notations introduced above, suppose that* $\mathfrak{h}$ *is endowed with an* Ad*-invariant inner product* $(,)$ *inducing the isomorphism* $\Psi : \mathfrak{h}^* \longrightarrow \mathfrak{h}$ *and let* $P : \mathfrak{h} \longrightarrow \mathfrak{h}_{i^*_{\mathfrak{h}}J(\dot{c})}$ *be the orthogonal projector onto* $\mathfrak{h}_{i^*_{\mathfrak{h}}J(\dot{c})}$. *Let* $\{u_i\}$ *be an orthonormal basis for* $\mathfrak{h}_{i^*_{\mathfrak{h}}J(\dot{c})}$ *with* $u_1 = \frac{\Psi(i^*_{\mathfrak{h}}J(\dot{c}))}{\left\|\Psi(i^*_{\mathfrak{h}}J(\dot{c}))\right\|}$. *Also, let the non-holonomic gauge* d_0^{NH} *be defined by the horizontal lift* (17). *Then, the corresponding dynamic phase equation becomes*

$$\frac{\mathrm{d}}{\mathrm{d}t} h_D h_D^{-1}(t) = \left(2K\left(\frac{\mathrm{d}}{\mathrm{d}t}c(t)\right) - 2K_{\mathrm{int}}(t)\right)\frac{\Psi(i^*_{\mathfrak{h}}J(\dot{c}))}{\left\|\Psi(i^*_{\mathfrak{h}}J(\dot{c}))\right\|^2} + \sum_{i=2}^{\dim \mathfrak{h}_{i^*_{\mathfrak{h}}J(\dot{c})}} (u_i, (I^{\mathfrak{h}}_{c(t)})^{-1}(i^*_{\mathfrak{h}}J(\dot{c})))u_i$$

$$h_D(t_1) = e.$$

In the above expression for the dynamic phase, (K_{int}) K denotes the (gauge-internal) kinetic energy of the controlled system in Q (see Appendix A). The above formula relates these physical quantities, which are directly involved in the system's dynamics, to the phases appearing during the full H-horizontally symmetric motion (see Corollary bellow).

Corollary. *Finally, if the solution* $\Pi(t) \in \mathfrak{g}^*$ *is such that* $i^*_{\mathfrak{h}}\Pi(t_1) = i^*_{\mathfrak{h}}\Pi(t_2)$ *then:*

- $g_G(t_2)$ *is the* holonomy *of the base path* $i^*_{\mathfrak{h}}\Pi(t)$ *in the* $H_{i^*_{\mathfrak{h}}J(\dot{c})}$*-bundle* $H \xrightarrow{\pi} O^H_{i^*_{\mathfrak{h}}J(\dot{c})}$ *with respect to the connection defined by* P *measured from* $g_G(t_1) = e$.
- *the solution for the constrained and controlled system* $c(t) \in Q$ *satisfies the following* phase relation *at time* t_2*:*

  $$\boxed{c(t_2) = h_D(t_2) g_G(t_2) \cdot d_0^{\mathrm{NH}}(t_2)}$$

  where $d_0^{\mathrm{NH}}(t_2)$ *is the horizontal lift of* $\tilde{c}(t)$ *with respect to the non-holonomic connection* [2], *starting from* $d_0(t_1) = c(t_1)$.
- *when, in addition, the base curve* $\tilde{c}(t) \in Q/G$ *is closed for* $t \in [t_1, t_2]$, *so* $\tilde{c}(t_1) = \tilde{c}(t_2)$, *then* $d_0(t_2) = g_G^{\mathrm{NH}} \cdot d_0(t_1)$ *where* g_G^{NH} *is the holonomy of the base path* $\tilde{c}$ *with respect to the non-holonomic connection in the bundle* $Q \longrightarrow Q/G$ *measured from the initial condition* $d_0(t_1) = c(t_1)$. *So, in this case,*

  $$\boxed{c(t_2) = h_D(t_2) g_G(t_2) \cdot g_G^{\mathrm{NH}} \cdot c(t_1)}.$$

4.5. Phases for systems with dipolar-magnetic-torque type of affine constraints

An interesting special case of affine-constrained systems which do not satisfy hypothesis (ii) of Section 3.2 but present *reconstruction phase formulas* is the following.

(ii′) The affine constraints are of *external dipolar-magnetic-torque* form, this is,

$$\mathfrak{A}^{\mathrm{Mech}}_{q(t)}(\dot{q}(t)) = I^{-1}_{q(t)} Ad^*_{h_M(t)} \hat{L}_0$$

for $\mathfrak{A}^{\mathrm{Mech}}$ denoting the *mechanical connection* (see [9]). Equivalently, the affine constraint can be put in the form

$$J(\dot{q}(t)) = Ad^*_{h_M(t)} \hat{L}_0$$

for some given curve $h_M(t) \in G$, with $h_M(t_1) = e$ and the initial momentum value $\hat{L}_0 \neq 0 \in \mathfrak{g}^*$.

The time derivative of the above equation is equivalent to the following *non-conservation of momentum equation*

$$\frac{\mathrm{d}}{\mathrm{d}t} J(\dot{q}(t)) = ad^*_{\dot{h}_M h_M^{-1}} J(\dot{q}(t))$$

where the right-hand side represents a *generalized torque* of a very special kind. In Section 5.4, we shall study the motion of a body with dipolar magnetic moment in an external magnetic field which can be described as a system with affine constraints of type (ii′) above. This justifies our terminology.

So we now assume (ii′) to hold and that we have a base-controlled curve $\tilde{c}(t)$. Next, we choose the *mechanical gauge* $d_0^{\mathrm{Mec}}(t)$ (18) because D for the above connection form $\mathfrak{A}^{\mathrm{Mech}}_{q(t)}$ is exactly the horizontal space with respect to the mechanical connection. Since constraints represent $\dim G$ equations, they fully characterize the dynamics of the group unknown $g(t)$ in $c(t) = g(t) \cdot d_0^{\mathrm{Mec}}(t)$. Indeed, D defines a principal connection, thus equations of motion (12) are trivial. These constraint equations in (ii′) can be written as

$$Ad^*_{h_M^{-1}(t)} Ad^*_{g(t)} I_{d_0^{\mathrm{Mec}}(t)} \left(g^{-1}\dot{g}\right) = \hat{L}_0 = \text{const.}$$

From this, we see that if we call $R_M(t) := h_M^{-1}(t)g(t) \in G$ and $\Pi(t) := I_{d_0^{\mathrm{Mec}}(t)}(g^{-1}\dot{g})$, then

$$Ad^*_{R_M(t)} \Pi(t) = \hat{L}_0 \tag{49}$$

so $\Pi(t) \in O_{L_0} \subset \mathfrak{g}^*$, the coadjoint orbit through $\hat{L}_0$, for all t. The corresponding equation giving the dynamics of $\Pi(t)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\Pi(t) = -ad^*_{R_M^{-1}\dot{R}_M}\Pi(t) = -ad^*_{(I^{-1}_{d_0^{\mathrm{Mec}}(t)}\Pi(t) - Ad_{g^{-1}}\dot{h}_M h_M^{-1})}\Pi(t).$$

Note that this equation is coupled to the one that defines $\Pi(t)$ from $g(t)$. Nevertheless, Eq. (49) implies that we are in the situation described in Appendix B and we can thus apply the reconstruction procedure [6] on the principal $G_{\hat{L}_0}$-bundle $G \simeq L^{-1}(\hat{L}_0) \longrightarrow O_{\hat{L}_0}$ to obtain $R_M(t)$ from a solution $\Pi(t) \in O_{\hat{L}_0}$. This yields the phase formula $R_M(t) = R_M^{\mathrm{Dyn}}(t) R_M^{\mathrm{Geom}}(t)$ where the dynamic phase $R_M^{\mathrm{Dyn}}(t)$ lies in $G_{\hat{L}_0}$ and $R_M^{\mathrm{Geom}}(t)$ is a horizontal lift of $\Pi(t)$ with respect to some chosen P-connection A_P in the $G_{\hat{L}_0}$-bundle $L^{-1}(\hat{L}_0) \simeq G \longrightarrow O_{\hat{L}_0}$. In this case, the dynamic phase equation, when put in terms of the original $g(t)$, reads

$$\frac{\mathrm{d}}{\mathrm{d}t} R_M^{\mathrm{Dyn}} R_M^{Dyn\;-1}(t) = A_P\left(\frac{\mathrm{d}}{\mathrm{d}t} R_M(t)\right)_g \tag{50}$$

$$= P\left(Ad_{h_M^{-1}}\left(I^{-1}_{c(t)} J(\dot{c})_{(t)} - \dot{h}_M h_M^{-1}\right)\right) \tag{51}$$

$$R_M^{\mathrm{Dyn}}(t_1) = id. \tag{52}$$

In Section 5.4, we shall work out the details of the above reconstruction formula in the magnetic dipole example.

Finally, if $\Pi(t_1) = \Pi(t_2)$ we then have a *phase formula* which fully characterizes the motion of the system $c(t) \in Q$ at time t_2:

$$c(t_2) = h_M(t_2) \cdot R_M^{\mathrm{Dyn}}(t_2) \cdot \mathrm{Hol}^P_{\Pi(t_{1,2})} \cdot d_0^{\mathrm{Mec}}(t_2)$$

where $\mathrm{Hol}^P_{\Pi(t_{1,2})}$ is the holonomy of the curve $\Pi(t)$ with respect to the P-connection in the bundle $L^{-1}(\hat{L}_0) \simeq G \longrightarrow O_{\hat{L}_0}$ measured from the initial value $e \in G$.

5. Examples

Here we illustrate our general considerations on simple examples of *base-controlled*, D-constrained systems. Examples of shape-controlled self deforming bodies with conserved angular momentum can be found in [3].

5.1. Vertical rotating disk

We consider the vertical rotating disk example from [2]. This gives an example of the systems considered in Section 3.3. In this case, $Q = \mathbb{R}^2 \times S^1 \times S^1 \ni q = (x, y, \theta, \varphi)$ and we consider $G = \mathbb{R}^2 \times S^1 \ni g = (x, y, \theta)$ (left) acting on itself. The Lagrangian reads

$$L(\dot{x}, \dot{y}, \dot{\theta}, \dot{\varphi}) = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} J \dot{\varphi}^2$$

and the non-holonomic constraints (non-sliding) are given by

$$\dot{x} = R \cos\varphi\, \dot{\theta}$$
$$\dot{y} = R \sin\varphi\, \dot{\theta}$$

where R is the radius of the disk. In this case, the base-controlled curve is $\tilde{c}(t) = \varphi(t)$, which represents the orientation of the vertical plane containing the disk, and

$$d_0^{\mathrm{NH}}(t) = (x_0, y_0, \theta_0, \varphi(t))$$

gives a non-holonomic gauge (which, in this example, also coincides with the mechanical gauge). Also,

$$\mathfrak{g}^q = \mathrm{span}\{(R\cos\varphi, R\sin\varphi, 1) \in \mathrm{Lie}(G) = \mathbb{R}^2 \oplus \mathfrak{s}(1)\}$$

and

$$J(\dot{c}) = (m\, \dot{x}, m\, \dot{y}, I\, \dot{\theta}).$$

From Section 3.3, the constraint equation in terms of $J(\dot{c})$ for this non-holonomic gauge reads $I_0^{-1} J(\dot{c}) \in \mathfrak{g}^q$, or,

$$J(\dot{c}) = \lambda(t)(mR\cos\varphi(t), mR\sin\varphi(t), I)$$

for some $\lambda(t) \in \mathbb{R}$ to be determined by the corresponding equation of motion (26) for $J(\dot{c})$:

$$\left\langle i^*_{d_0(t)} \left(\frac{\mathrm{d}}{\mathrm{d}t} J(\dot{c}) \right), (R\cos\varphi, R\sin\varphi, 1) \right\rangle = 0$$
$$\dot{\lambda}(mR^2 + I) + \lambda \left[\frac{\mathrm{d}}{\mathrm{d}t} (mR\cos\varphi, mR\sin\varphi, I) \right] \cdot (R\cos\varphi, R\sin\varphi, 1) = 0.$$

Note that the second term in the last equation is zero because the two vectors are orthogonal. Then, since $\lambda(t) = \dot{\theta}$ by the definition of the momentum $J(\dot{c})$, we have

$$\dot{\lambda}(mR^2 + I) = \ddot{\theta}(mR^2 + I) = 0$$

which is the vertical equation of motion derived in [2]. The above conservation law can be directly computed via equation (31) since $\gamma_1^1 = 0$ (the underlying linear connection in the 1-dimensional bundle $\mathfrak{g}^D \longrightarrow Q/G = S^1$ is flat, see Section 3.3). Consequently, $\dot{\theta}$ is constant. Finally, since we have solved for $J(\dot{c})$ using the equation of motion and of constraints, we can apply formula (30) obtaining

$$g(t) = \left(\dot{\theta}\, mR \left(\int_{t_1}^{t} \mathrm{d}s \cos\varphi(s) \right), \dot{\theta}\, mR \left(\int_{t_1}^{t} \mathrm{d}s \cos\varphi(s) \right), I\, \dot{\theta}(t - t_1) \right).$$

Note that $g_{\mathrm{Mech}}(t)=(0,0,0)$ in this case. At last, the full solution $c(t)\in Q$ is

$$\begin{aligned} c(t) &= g(t)\cdot d_0(t)\\ &= \left(\dot\theta\, mR\left(\int_{t_1}^t \mathrm{d}s\cos\varphi(s)\right), \dot\theta\, mR\left(\int_{t_1}^t \mathrm{d}s\cos\varphi(s)\right), I\,\dot\theta(t-t_1)\right)\cdot(x_0,y_0,\theta_0,\varphi(t))\\ &= \left(\dot\theta\, mR\left(\int_{t_1}^t \mathrm{d}s\cos\varphi(s)\right)+x_0, \dot\theta\, mR\left(\int_{t_1}^t \mathrm{d}s\cos\varphi(s)\right)+y_0, I\,\dot\theta(t-t_1)+\theta_0,\varphi(t)\right)\end{aligned}$$

from which we clearly see that motion is induced on the group variables from the base-controlled curve $\varphi(t)$ due to the presence of the non-sliding non-holonomic (D-)constraints.

5.2. Ball on a rotating turntable

We also recall the setting for describing a ball on a rotating turntable from [2]. This is an example of the systems considered in Sections 3.2 and 3.4. The corresponding Lagrangian on $Q=\mathbb{R}^2\times SO(3)$ is

$$L=\frac{1}{2}m(\dot x^2+\dot y^2)+\frac{1}{2}mk^2(\omega_x^2+\omega_y^2+\omega_z^2),$$

and the non-sliding *affine D-constraints* for the ball motion are

$$\begin{aligned} -\dot x+a\omega_y &= \Omega y\\ \dot y+a\omega_x &= \Omega x\end{aligned}$$

where $(x,y)\in\mathbb{R}^2$ denote the ball's position and $\dot g=\omega_x\xi_x^R(g)+\omega_y\xi_y^R(g)+\omega_z\xi_z^R(g)$ the angular velocity of $g(t)\in SO(3)$ representing the ball's rotation around its center. Here, $\xi_i^R(g)$ denotes the right invariant vector in $T_gSO(3)$ whose value at e is $\xi_i\in\mathfrak{so}(3)$, the generator of rotations about the i-axis. Also above, a is the ball's radius, mk^2 its (any) principal moment of inertia and Ω the given angular velocity of the rotating turntable. To take these equations to the form of Eq. (23) we define

$$\begin{aligned} \mathfrak{A}^D_{(x,y,g)}(\dot x,\dot y,\dot g) &= \left((\dot x,\dot y,\dot g),v_q^4\right)\frac{v_q^4}{\|v_q^4\|^2}+\left((\dot x,\dot y,\dot g),v_q^5\right)\frac{v_q^5}{\|v_q^5\|^2}\\ \gamma(x,y,g) &= \Omega y\frac{v_q^4}{\|v_q^4\|^2}+\Omega x\frac{v_q^5}{\|v_q^5\|^2}\end{aligned}$$

where $(,)=(,)_{\mathbb{R}^2}+(,)_{\mathfrak{so}(3)}$ denotes the kinetic energy inner product on $Q=\mathbb{R}^2\times G$ with $G=SO(3)$ and $v_q^4=-\frac{\partial}{\partial x}+a\xi_y^R(g)$, $v^5=\frac{\partial}{\partial y}+a\xi_x^R(g)$ in TQ. Note that both $D:=\mathrm{Ker}(A^D)=\mathrm{Span}\{a\frac{\partial}{\partial x}+\xi_y^R(g)\,;-a\frac{\partial}{\partial y}+\xi_x^R(g)\,;\xi_z^R(g)\}$ and γ are G-invariant for the natural *right* action of G on Q. Also notice that on the previous sections we considered a *left* G action on Q, so we turn the above natural right action into a left one by defining $g\cdot(x,y,h)=(x,y,hg^{-1})$ in $\mathbb{R}^2\times G$.

In this case, since shape space B is $\mathbb{R}^2$, the controlled curve $\tilde c(t)=(x(t),y(t))$ represents the position of the contact point between the ball and the table as describing a given trajectory. So the problem is to find out how the ball rotates (i.e. to find $g(t)$) due to the presence of the non-sliding affine constraints and to the fact that the contact point is *moving in this known way* $(x(t),y(t))$. From Section 3.2, we know that the corresponding equations for the unknown $g(t)\in G$ are the equations of motion (12) and the constraint equation (24). Also from that section, we know that we can simplify the constraint equation by considering an *affine gauge* $d_0^{\mathrm{Aff}}(t)$ satisfying (25). In the present example, $\mathfrak{g}^{(x,y,g)}=\mathrm{Span}\{Ad_{g^{-1}}\xi_z\}$ with $\xi_z\in\mathfrak{so}(3)$ the generator of rotations about the z-axis. Also, the momentum map for the above G-symmetric Lagrangian is $J(\dot x,\dot y,\dot g)=-mk^2g^{-1}\dot g\in\mathfrak{so}(3)\simeq\mathfrak{so}^*(3)$. One possible affine gauge choice is

$$d_0^{\mathrm{Aff}}(t)=(x(t),y(t),g_{\mathrm{Aff}}(t))$$

with $\dot g_{\mathrm{Aff}}g_{\mathrm{Aff}}^{-1}=\frac{1}{a}(-\dot y+\Omega x)\,\xi_x+\frac{1}{a}(\dot x+\Omega y)\,\xi_y$, i.e., with no z-(spatial) angular velocity component. Consequently, the full solution $c(t)=(x(t),y(t),g_{\mathrm{tot}}(t))$ is written as

$$c(t) = g(t) \cdot d_0^{\mathrm{Aff}}(t) = (x(t), y(t), g_{\mathrm{Aff}}(t) g^{-1}(t))$$

with $g(t)$ satisfying:

(1) (*Constraints*) $g^{-1}\dot{g} \in \mathfrak{g}^{d_0^{\mathrm{Aff}}(t)} = \mathrm{Span}\{Ad_{g_{\mathrm{Aff}}(t)^{-1}}\xi_z\}$

(2) (*Motion*) $\left(\frac{\mathrm{d}}{\mathrm{d}t} J(\dot{x}, \dot{y}, \dot{g}), Ad_{(g_{\mathrm{Aff}}(t)g^{-1}(t))^{-1}}\xi_z\right)_{\mathfrak{so}(3)} = 0.$

It is easy to see that, by calling $g_{\mathrm{tot}}(t) = g_{\mathrm{Aff}}(t)g^{-1}(t)$, Eq. (2) above reduces to $J_z^S(\dot{c}) := mk^2 \left(\dot{g}_{\mathrm{tot}} g_{\mathrm{tot}}^{-1}, \xi_z\right)_{\mathfrak{so}(3)} =$ const., i.e. the z-component of the (spatial) angular momentum is conserved, since

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}t} J(\dot{x}, \dot{y}, \dot{g}), Ad_{g_{\mathrm{tot}}^{-1}}\xi_z\right)_{\mathfrak{so}(3)} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(J(\dot{x}, \dot{y}, \dot{g}), Ad_{g_{\mathrm{tot}}^{-1}}\xi_z\right)_{\mathfrak{so}(3)} + \left(J(\dot{x}, \dot{y}, \dot{g}), Ad_{g_{\mathrm{tot}}^{-1}} ad_{\dot{g}_{\mathrm{tot}} g_{\mathrm{tot}}^{-1}}\xi_z\right)_{\mathfrak{so}(3)} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} J_z^S(\dot{c}) + mk^2 \left(g_{\mathrm{tot}}^{-1}\dot{g}_{\mathrm{tot}}, Ad_{g_{\mathrm{tot}}^{-1}} ad_{\dot{g}_{\mathrm{tot}} g_{\mathrm{tot}}^{-1}}\xi_z\right)_{\mathfrak{so}(3)}\end{aligned}$$

and the second term in the r.h.s. above vanishes. Notice that, although we have a conservation law, it is a 1-dimensional one and no non-trivial reconstruction phase formulas for $g(t)$ follow from it.

Remark 5.1 (*Conservation due to Symmetry*). Using Remark 2.6, we can easily see, by considering G as being only rotations about the z-axis and acting by *left multiplication* on Q, that Eq. (11) becomes directly the above z-component conservation of the corresponding (spatial) angular momentum. Nevertheless, notice that this setting does not give any insight on the constraint-base-induced motion $g(t)$.

Now, from (1) above, we get

$$g^{-1}\dot{g} = Ad_{g_{\mathrm{Aff}}^{-1}(t)}\omega_z\xi_z$$

and from (2) that

$$\omega_z = \text{const.}$$

So, finally, the full base-induced group variable $g_{\mathrm{tot}}(t)$ in the full system's motion $c(t)$ is obtained as a product of the two simpler factors $g_{\mathrm{Aff}}(t)g^{-1}(t)$ described above. Note that, in this simple example, the factorization result we obtained following our general considerations is the same as what we obtain by proposing the solution $g_{\mathrm{tot}}(t) = g_{\mathrm{Aff}}(t)g^{-1}(t)$ for the constraints plus conservation equations as expressed in Ref. [2]:

$$\dot{g}_{\mathrm{tot}} = \frac{1}{a}(-\dot{y} + \Omega x)\,\xi_x^R(g_{\mathrm{tot}}) + \frac{1}{a}(\dot{x} + \Omega y)\,\xi_y^R(g_{\mathrm{tot}}) + (\mathrm{const})\,\xi_z^R(g_{\mathrm{tot}}).$$

5.3. A non-holonomically constrained self-deforming body

This is an example of a base controlled and D-constrained system presenting *phase formulas* due to (non-full) horizontal symmetries (Section 4.4, [2]). The system consists of two rigid spheres as in Fig. 1. The small ball is attached to the inside of the big one (*holonomic* constraint) which, in turn, can move freely. The key ingredient is that the first rotates *without sliding with respect to the second*. This last requirement represents a non-holonomic D-constraint on the total system and we further assume that no external forces are present. This gives a simplified model for a small robot (the small ball) moving inside a space-craft (the big ball). As we shall see below, this example generalizes the treatment of [3] by allowing *non-holonomic constraints* to induce total body motion from arbitrarily controlled (base) variables living in a smaller space within the usual shape space $Q/SO(3)$.

The configuration space is $Q = SO(3) \times S_r^2 \times SO(3) \ni (R_1, r_2, R_3)$ defined by requiring $r_i(t) = R_i(t)r_{io} \in \mathbb{R}^3$ to be the position of the point i with respect to a reference system with axes parallel to those of a chosen inertial one and with origin in the corresponding ball's center (see Fig. 1). We denoted by S_r^2 the 2-sphere of radius $r = \|r_2(t)\| =$ const. In this coordinates, the Lagrangian takes the simple kinetic energy form

$$L(\dot{R}_i) = T(\dot{R}_i) = \frac{1}{2}\left(R_1^{-1}\dot{R}_1, I_1 R_1^{-1}\dot{R}_1\right)_{\mathfrak{so}(3)} + \frac{1}{2}\mu r^2 (\dot{r}_2 \cdot \dot{r}_2) + \frac{1}{2}\left(R_3^{-1}\dot{R}_3, I_3 R_3^{-1}\dot{R}_3\right)_{\mathfrak{so}(3)}$$

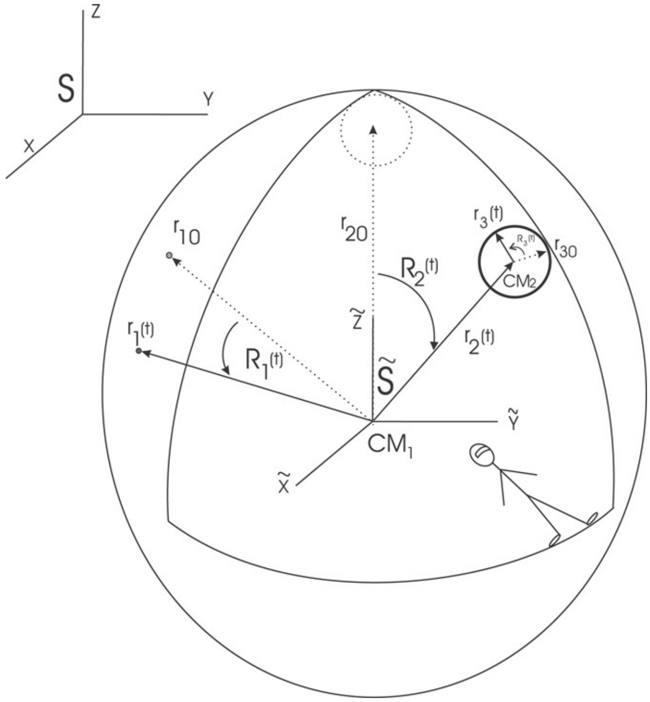

Fig. 1. The big ball's rotation $R_1(t)$ and the position of the center CM_2 of the small ball, both as seen from reference system $\tilde{S}$, are described by $r_1(t) = R_1(t)r_{10}$ and $r_2(t) = R_2(t)r_{20}$, respectively. $\tilde{S}$ has its origin at the center CM_1 of the big ball and axes parallel to those of an inertial frame S. The rotation $R_3(t)$ of the small ball about its center CM_2 is described by the vector $r_3(t) = R_3(t)r_{30}$.

and the 2 non-sliding non-holonomic D-constraint equations (for $r_{20} = r\check{z}$) read

$$\begin{aligned} -\frac{a}{r}\left(Ad_{R_2^{-1}}\left(-\dot{R}_1R_1^{-1} + \dot{R}_3R_3^{-1}\right), \xi_x\right)_{\mathfrak{so}(3)} &= \left(Ad_{R_2^{-1}}\left(-\dot{R}_1R_1^{-1} + \dot{R}_2R_2^{-1}\right), \xi_x\right)_{\mathfrak{so}(3)} \\ -\frac{a}{r}\left(Ad_{R_2^{-1}}\left(-\dot{R}_1R_1^{-1} + \dot{R}_3R_3^{-1}\right), \xi_y\right)_{\mathfrak{so}(3)} &= \left(Ad_{R_2^{-1}}\left(-\dot{R}_1R_1^{-1} + \dot{R}_2R_2^{-1}\right), \xi_y\right)_{\mathfrak{so}(3)}. \end{aligned} \tag{53}$$

Above, $I_1 = \mathrm{diag}(\frac{2}{5}m_1(r+a)^2)$, $I_3 = \mathrm{diag}(\frac{2}{5}m_2a^2)$, with a the small ball's radius, are the inertia tensors of the balls with respect to their respective centers in the standard basis $\{\xi_i\}$ of $\mathfrak{so}(3)$ formed by the generators of i-axis rotations, $i = x, y, z$, and $\mu = \frac{m_1m_2}{m_1+m_2}$.

Remark 5.2 (R_2 *Expressions*). It can be easily seen that every expression depending on R_2 given within this subsection, like the constraints above, is invariant under $R_2(t) \rightsquigarrow R_2(t)R_z(t)$ with $R_z(t)$ a rotation about the z-axis. This means that they really depend on $r_2(t) = r\ R_2(t)\check{z}$, but we keep the rotational dependence for simplicity. Given $r_2(t) \in S_r^2$, one choice for $R_2(t)$ is given by the horizontal lift in the $U_z(1)$ bundle $SO(3) \longrightarrow S_r^2$ (see [3]).

The distribution $D \subset TQ$ of tangent vectors satisfying Eqs. (53) has dimension $\dim D = \dim Q - 2 = 6$.

Now, consider the group $G = SO(3)^2 \ni (R, g_3)$ (left) acting on Q via

$$(R, g_3)\cdot(R_1, r_2, R_3) = (RR_1, Rr_2, RR_3g_3^{-1}).$$

It is easy to see that both L and D are G-invariant. Shape space $B = Q/G$ can be parameterized by elements $r_{2,1} \in S_r^2$ and hypothesis $(H1)$ of Section 2.1 is satisfied. We also assume $(H2)$ to hold, which, in this case, means that the controlled part of the motion is represented by a gauge curve $d_0(t) = (e, r_{2,1}(t), e)$. If $c(t) = (R_1(t), R_2(t), R_3(t)) \in Q$ represents the full system's motion, then $r_{2,1}(t) = R_1^{-1}(t)r_2(t)$ represents the position of the CM_2 as seen from a reference system with origin at CM_1 and with axes rotating with R_1, i.e. a system rotating with the big ball. Indeed, the full motion can be written as $c(t) = (R_1(t), R_3^{-1}(t)R_1(t))\cdot d_0(t)$ and note that *no constraints* remain on the *controlled variable* $r_{2,1}$ (it can be arbitrarily chosen within $B \simeq S_r^2$). Also, notice that from the dim $Q = 8$ variables, as 2 are being freely controlled, we are left with 4 *equations of motion* plus the 2 *D-constraints* to solve.

More physically, the problem is to find the *total reorientation of the system* $R_1(t)$ induced by the *inside motion*. This, in turn, is generated by the (known) inner translational motion $d_0(t)$ of the small ball and followed by its D-induced rotational motion $R_1^{-1}(t)R_3(t)$, both as seen from a system fixed to the big ball, due to the presence of the non-sliding constraints and fulfilling the corresponding Lagrangian equations of motion.

Remark 5.3 (*Measurement of* $r_{2,1}$). The curve $r_{2,1}(t)$ is the one that an astronaut standing in the space-craft, modeled by the big ball, would see as the small ball's center moves (see Fig. 1). Consequently, it can also be measured in lab conditions, when the space-craft is attached to the floor (and cannot rotate), but when the small ball rehearses the same translational motion $r_{2,1}$ that will occur in space. Notice that this cannot be done with the rotational motion of the small ball since it must obey an additional equation of motion ((54) below) which is not invariant under rotating $R_1(t)$-reference frame transformations.

We now turn to the equations of motion. Consider the subgroup $H := \{(R, e), R \in SO(3)\} \subset G$. It can be easily checked that $\mathfrak{h}_Q = (\mathrm{Lie}(H))_Q \subset D$ and that, for $q = (R_1, r_2, R_3) \in Q$,

$$\mathfrak{g}^q = \mathfrak{h} \oplus \mathrm{Span}\{Ad_{R_3^{-1}} Ad_{R_2} \xi_z^3\}$$

with ξ_z^3 seen as an element of the second $\mathfrak{so}(3)$ copy in $\mathrm{Lie}(G) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$. The above means that we are in the presence of *non-full* $\mathfrak{h}$-horizontal symmetries [2]. Consequently,

$$i_{\mathfrak{h}}^* J(\dot{c}) = I_1 \dot{R}_1 R_1^{-1} + \mu r^2 (r_2 \times \dot{r}_2)^{\curlywedge} + I_3 \dot{R}_3 R_3^{-1} = I_1 \dot{R}_1 R_1^{-1} + Ad_{R_2} I_{20} R_2^{-1} \dot{R}_2 + I_3 \dot{R}_3 R_3^{-1}$$

in $\mathfrak{so}(3) \overset{\text{metric}}{\simeq} \mathfrak{so}^*(3) = \mathrm{Lie}(H)^*$ is a *conserved quantity*. Above, $^{\curlywedge}$ denotes the (Lie algebra) isomorphism $\mathbb{R}^3 \longrightarrow \mathfrak{so}(3)$ and

$$I_{2,0} = \mu r^2 \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}.$$

This horizontal momentum represents the *total angular momentum* of the system [5].

Remark 5.4 (*Relevance of the Present Approach due to Constraints*). We would like to remark that, if we considered only H as symmetry group, as it is done for non-constrained self deforming bodies (see [3]), then the D-constraints are no longer vertical (Remark 2.2). In other words, the corresponding base variables ($r_{2,1}$ and $R_{3,1}$) become constrained and it would make no sense to think of them as *arbitrarily* controlled or given. By considering the bigger G instead, we restrict to the smaller base-variables space which are actually a priori arbitrarily controllable.

Note that dim $\mathfrak{g}^q = 4$, so the above conservation law represents only 3 of equations of motion (11). The remaining equation is

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} \left(R_3^{-1} \dot{R}_3 \right), Ad_{R_3^{-1}} Ad_{R_2} \xi_z \right)_{\mathfrak{so}(3)} = 0, \tag{54}$$

which states that there is no angular acceleration of the smaller ball rotation in the CM_1–CM_2 direction. This same effect is observed in the ball on a rotating turntable example (see [2] and the previous section).

Finally, from Section 4.4, we know that we can write (reconstruction) phase formulas for the system's motion due to the horizontal conservations. Below, we summarize the Q-reconstruction procedure for obtaining the solution $c(t)$ from the base motion $\tilde{c}(t)$.

- We start with $d_0(t) = (e, r_{2,1}(t), e)$, and $c(t) = (R_1(t), r_2(t), R_3(t)) \in Q$ representing the desired solution.
- To use the results of the previous sections, we choose a *non-holonomic gauge* d_0^{NH}. We fix it by $d_0^{\mathrm{NH}}(t) = (R_{1,NH}, R_{3,NH}^{-1} R_{1,NH})(t) \cdot d_0(t)$ with

$$[\text{constraints} + 1\text{eq.}] \quad -\frac{a}{r} \left(-\dot{R}_{1,NH} R_{1,NH}^{-1} + \dot{R}_{3,NH} R_{3,NH}^{-1} \right) = Ad_{R_{1,NH}} \dot{R}_{2,1} R_{2,1}^{-1}$$

$$\left[i_{\mathfrak{h}}^{*}J(\dot{d}_0^{\mathrm{NH}})=0\right]\; I_3\dot{R}_{3,NH}R_{3,NH}^{-1}+\left(I_1+Ad_{R_{1,NH}R_{2,1}}I_{20}Ad_{R_{2,1}^{-1}}\right)\dot{R}_{1,NH}R_{1,NH}^{-1}$$
$$+Ad_{R_{1,NH}R_{2,1}}I_{20}R_{2,1}^{-1}\dot{R}_{2,1}=0$$

with $r_{2,1}(t)=rR_{2,1}(t)\check{z}$ and trivial initial conditions for $R_{i,NH}$, $i=1,3$. Equivalently, we could have chosen the *horizontal* non-holonomic gauge (17) leading to the same $i_{\mathfrak{h}}^{*}J(\dot{d}_0^{\mathrm{NH}})=0$ equation plus constraint equation (53) plus one more (involved) equation.

- We now write $c(t)=(R,g_3)(t)\cdot d_0^{\mathrm{NH}}(t)$. Notice that, since the horizontal symmetries are non-full, Eq. (16) for $g(t)\equiv(R,g_3)(t)$ is non-trivial and yields

$$g_3^{-1}\dot{g}_3=\lambda(t)\;Ad_{R_{3,NH}^{-1}}Ad_{R_{1,NH}R_{2,1}}\xi_z$$

with $\lambda(t)\in\mathbb{R}$ to be determined. The corresponding vertical equations of motion for $g(t)$ read

$$\begin{aligned}&[\mathfrak{h}\text{-conservation}]\; i_{\mathfrak{h}}^{*}J(\dot{c})=\text{const}=Ad_R\left(I^{\mathfrak{h}}_{d_0^{\mathrm{NH}}}R^{-1}\dot{R}-\lambda I_3Ad_{R_{1,NH}R_{2,1}}\xi_z\right)=:Ad_R\Pi^{\mathfrak{h}}(t)\\&[\text{Eq. (54)}]\;\dot{\lambda}=\left(\frac{\mathrm{d}}{\mathrm{d}t}\left[Ad_{R_{3,NH}^{-1}}R^{-1}\dot{R}+R_{3,NH}^{-1}\dot{R}_{3,NH}\right],Ad_{R_{3,NH}^{-1}}Ad_{R_{1,NH}R_{2,1}}\xi_z\right)_{\mathfrak{so}(3)}\end{aligned}\qquad(55)$$

with $I^{\mathfrak{h}}_{d_0^{\mathrm{NH}}}=I_1+Ad_{R_{1,NH}R_{2,1}}I_{2,0}Ad_{(R_{1,NH}R_{2,1})^{-1}}+I_3:\mathfrak{h}\longrightarrow\mathfrak{h}\simeq\mathfrak{h}^{*}\simeq\mathfrak{so}(3)$ the corresponding restricted inertia tensor.

Above, g_3 is s.t. $R_{1,NH}^{-1}R_{3,NH}g_3^{-1}=R_{3,1}(t)=R_1^{-1}(t)R_3(t)$ represents the rotational motion of the small ball as seen from a reference system with origin at CM_2 and axes rotating with the big ball, i.e., is what an astronaut standing inside the big ball would see (see Remark 5.3 and Fig. 1). Also, $\lambda=(g_3^{-1}\dot{g}_3,Ad_{R_{3,NH}^{-1}}Ad_{R_{1,NH}R_{2,1}}\xi_z)_{\mathfrak{so}(3)}=(\dot{g}_3g_3^{-1},Ad_{R_3^{-1}R_2}\xi_z)_{\mathfrak{so}(3)}$ represents a dynamical correction to the (spatial) angular velocity of the small ball in the direction CM_1–CM_2 needed for Eq. (54) to be satisfied from an inertial reference frame.

Notice that the above equations of motion for R and λ are coupled. Nevertheless, in the obtained factorization

$$c(t)=\left(R\;R_{1,NH},\,R\;R_{1,NH}\;R_{2,1},\,R\;R_{3,NH}\;g_3^{-1}\right)$$

every element as defined above represents a simpler piece from which the overall motion is constructively induced from the known one $R_{2,1}(t)$ on the base. This shows how we can (geometrically) take advantage of the kinematical structure of the system for writing the controlled solution. Moreover, the global reorientation R can be further factorized by implementing the *phase formulas* corresponding to the $\mathfrak{h}$- conservation reconstruction (Section 4.4).

The phase formula for R. From Section 4.4, we know that $R(t)$ can be reconstructed from the *body total angular momentum* $\Pi^{\mathfrak{h}}(t)$ solution on $O_{i_{\mathfrak{h}}^{*}J(\dot{c})}\simeq S^2_{\mathrm{radius}=i_{\mathfrak{h}}^{*}J(\dot{c})}\subset\mathrm{Lie}(H)=\mathfrak{so}(3)\simeq\mathbb{R}^3$ within the $U(1)$-bundle $SO(3)\longrightarrow S^2_{\mathrm{radius}=\|i_{\mathfrak{h}}^{*}J\|}$ (see details in [3]). In this case, $\Pi^{\mathfrak{h}}(t)$ is given by Eq. (55) and, from (45) via $\mathfrak{so}(3)\simeq\mathbb{R}^3$,

$$\dot{\Pi}^{\mathfrak{h}}(t)=\Pi^{\mathfrak{h}}(t)\times\left((I^{\mathfrak{h}}_{d_0^{\mathrm{NH}}(t)})^{-1}\left[\Pi^{\mathfrak{h}}(t)+\lambda I_3R_{1,NH}\frac{r_{2,1}}{r}\right]\right)$$

with $\times$ standing for the usual vector product in $\mathbb{R}^3$. This equation coincides with the one generically presented in [3] but in a *very precise non-holonomic gauge*, $\tilde{d}_0^{\mathrm{NH}}=(e,g_3)\cdot d_0^{\mathrm{NH}}$, which makes the whole procedure compatible with the D-constraints. Also in this case, this equation appears coupled to another equation, i.e. that of $\lambda(t)$, since the horizontal symmetries are non-full.

The *phase formula* corresponding to the reconstruction of Section 4.4, for $i_{\mathfrak{h}}^{*}J\neq 0$, reads

$$R(t)=\exp\left(\theta^{\mathrm{Dyn}}(t)\frac{i_{\mathfrak{h}}^{*}J}{\|i_{\mathfrak{h}}^{*}J\|}\right)R_1^{\mathrm{Geom}}(t)$$

with the constant $i_{\mathfrak{h}}^{*}J\in\mathfrak{so}(3)$. The *geometric phase* $R^{\mathrm{Geom}}(t)$ is the *horizontal lift* of the body total angular momentum curve $\Pi^{\mathfrak{h}}(t)$ in the $U(1)$-bundle $SO(3)\longrightarrow S^2_{\mathrm{radius}=\|i_{\mathfrak{h}}^{*}J\|}$ with respect to the connection $A_g(\dot{g})=$

$(\dot{g}g^{-1}, \frac{i^*_{\mathfrak{h}}J}{\|i^*_{\mathfrak{h}}J\|})_{\mathfrak{so}(3)}$ (for details, see [3]). The *dynamical phase* $\theta^{\mathrm{Dyn}}(t) \in U(1) = H_{i^*_{\mathfrak{h}}J}$ is defined by (recall Section 4.4)

$$\theta^{\mathrm{Dyn}}(t) = \frac{1}{\left\|i^*_{\mathfrak{h}}J\right\|}\int_{t_1}^{t} \mathrm{d}s \left[2K\left(\frac{\mathrm{d}}{\mathrm{d}t}c(s)\right) - 2K_{\mathrm{int}}(s) + \lambda(s)\left(\frac{2}{5}m_2a^2\right)\left(\xi_z, \left(I_e^{\mathfrak{h}}\right)^{-1} Ad_{R_2^{-1}(s)} i^*_{\mathfrak{h}}J\right)_{\mathfrak{so}(3)} + \frac{\lambda(s)^2\left(\frac{2}{5}m_2a^2\right)^2}{\frac{2}{5}m_1(r+a)^2 + \frac{2}{5}m_2a^2}\right] + \theta_0^{\mathrm{Dyn}}$$

where K represents the kinetic energy of the whole Q system given in Appendix A, $I_e^{\mathfrak{h}} = I_1 + I_{2,0} + I_3$ and rotation $R_2(t)$ gives the physical motion of CM_2 in $c(t)$. Notice the unavoidable (dynamical) λ dependence in the dynamical phase formula due to the fact that the horizontal symmetries are non-full (also compare to the non-D-constrained case of [3]).

Finally, it is worth noting that, when the solution $\Pi^{\mathfrak{h}}(t)$ is simple and closed for $t \in [t_1, t_2]$, then

$$R(t_2) = \exp\left(\left(\theta^{\mathrm{Dyn}}(t_2) + \theta^{\mathrm{Geom}}\right)\frac{i^*_{\mathfrak{h}}J}{\left\|i^*_{\mathfrak{h}}J\right\|}\right)R_1(t_1)$$

with $\theta^{\mathrm{Dyn}}(t_2)$ as given above and θ^{Geom} given (mod. 2π) by minus the (signed) *solid angle* enclosed by $\Pi^{\mathfrak{h}}(t)$ in the 2-sphere of radius $\|i^*_{\mathfrak{h}}J\|$ within $\mathbb{R}^3 \simeq \mathfrak{so}(3)$. The above is an example of a (D-)generalized self deforming body phase formula, not encoded in [3].

Remark 5.5 (*Control*). The above formulas can be useful for control purposes, this is, when you want to find the suitable base curve $R_{2,1}(t)$ inducing a certain desired global reorientation $R(t_2)$.

Remark 5.6 (*The Case* $i^*_{\mathfrak{h}}J = 0$). In this case, the equation for R is *geometrical*, meaning that it is a horizontal lift equation along $\tilde{d}_0^{\mathrm{NH}} = (e, g_3)\cdot d_0^{\mathrm{NH}}$ with respect to the $\mathfrak{h}$-mechanical connection. Nevertheless, this equation is coupled to that of g_3 (equiv. λ) which is not of geometric nature. Consequently, the complete motion induction from the initial *controlled* base variables $\tilde{c}(t) = R_{2,1}(t) \in B$ is not entirely geometrical. The cause is that horizontal symmetries are non-full (compare with Remark 4.11) and so they do not exhaust the whole vertical dynamics (i.e. because of the additional dynamical equation (54), see also [2] for similar comments).

5.4. Deforming body with dipolar magnetic moment in an external magnetic field

Here we describe the motion of a (deforming) body with magnetic moment $M \in \mathbb{R}^3$ in the presence of an external magnetic field. This system is modeled as an affine D-constrained and controlled system for which momentum is not conserved because of the magnetic applied forces and which is, thus, not covered by the analysis of [3]. We shall assume the following hypothesis about the magnetic nature of the system to hold:

- the magnetic moment is proportional to the *total angular momentum* J, i.e.

  $$M = \gamma J$$

  where γ is the *gyromagnetic ratio* [5].
- the interaction with an external magnetic field B is of dipolar type [5], this is

  $$\frac{\mathrm{d}}{\mathrm{d}t}J = M \times B$$

  where $M \times B$ is the external torque acting on the dipole and $\times$ denotes the standard vector product in $\mathbb{R}^3$.
- the above holds even when the shape $c(t) \in Q$ (see [9,11]) of the underlying body and the field $B(t)$ are changing with time.

From the above assumptions, the equation of motion for the angular momentum $J(\dot{c})$ of the body is

$$\frac{\mathrm{d}}{\mathrm{d}t}J(\dot{c}) = \gamma J(\dot{c}) \times B(t).$$

If we define the corresponding *Larmor frequency vector* [5] as $\omega_l(t) := -\gamma B(t) \in \mathbb{R}^3$, then the above can be re-expressed as

$$J(\dot{c}) = h_M(t)L_0 \overset{\Psi}{\equiv} Ad^*_{h_M(t)}\hat{L}_0$$

where $h_M(t) \in SO(3)$ is defined by

$$\dot{h}_M h_M^{-1}(t) = \hat{\omega}_l(t)$$
$$h_M(t_1) = Id$$

where $\hat{}$ denotes the (Lie algebra) isomorphism $\mathbb{R}^3 \longrightarrow \mathfrak{so}(3)$ and $\hat{L}_0$ denotes the initial value $J(\dot{c}(t_1))$ seen as an element of $so(3)^*$ through the usual isomorphisms.

The equations for the motion of such a system can be derived from the affine-constrained Lagrangian system $(TQ, \mathcal{L}, \mathcal{A}^D, \Gamma)$ where

- $Q \longrightarrow Q/G$ is the configuration space of the underlying deforming body [9,3] with symmetry group $G = SO(3)$,
- the Lagrangian is given by the kinetic energy contribution $\mathcal{L}(\dot{q}) = \frac{1}{2}k_q(\dot{q}, \dot{q})$, where k_q is a G-invariant metric on TQ induced by the standard $\mathbb{R}^3$-metric [9],
- $\mathcal{A}^D$ is the *mechanical* principal connection 1-form on $Q \longrightarrow Q/G$ given by

$$\mathcal{A}^D(\dot{q}) = I_q^{-1}J(\dot{q})$$

  with I_q giving the usual *inertia tensor* of the body and $J : TQ \longrightarrow \mathfrak{g}^*$ the usual *angular momentum* map,
- $\Gamma : Q \longrightarrow \mathfrak{g}$ is the map given by

$$\Gamma(q) = I_q^{-1}(Ad^*_{h_M(t)}\hat{L}_0).$$

The *affine constraints* for the physical curve $c(t)$ become

$$\mathcal{A}^D(\dot{c}(t)) = \Gamma(c(t)). \tag{56}$$

We now continue with the analysis in the *controlled case*, i.e., we add hypothesis $(H2)$ that the base curve $\tilde{c}(t) \in Q/G$, representing the changing body's shape, is given.

From Section 4.5, we know that D corresponds to the *mechanical connection* $\mathcal{A}^D$ and that equations of motion (12) for $g(t)$ are trivial. The only remaining D-constraint equations for $g(t)$ in a *mechanical gauge* $d_0^{\mathrm{Mec}}(t)$ (18) with $d_0^{\mathrm{Mec}}(t_1) = c(t_1)$, read

$$Ad^*_{g(t)}I^g_{d_0^{\mathrm{Mec}}(t)}\left(g^{-1}\dot{g}\right) = Ad^*_{h_M(t)}\hat{L}_0$$
$$g(0) = Id.$$

Also following Section 4.5, we call

$$R_M(t) = h_M^{-1}(t)g(t) \in SO(3)$$

and note that

$$Ad^*_{R_M(t)}I_{d_0^{\mathrm{Mec}}(t)}\left(g^{-1}\dot{g}\right) = \hat{L}_0$$

is a *conserved quantity*. The passage from g to R_M can be understood as passing to describe the system from a new reference frame which is rotating via $h_M(t)$ with respect to the original (inertial) frame, with spatial angular velocity $\omega_l(t)$ (see [5] pp. 231).

Then, following Appendix B once more, the rotation $R_M(t)$ can be reconstructed from the *body angular momentum* (5) $\Pi(t) = I_{d_0^{\mathrm{Mec}}(t)}\left(g^{-1}\dot{g}\right)$ within the $U(1)$-bundle $L^{-1}(\hat{L}_0) \simeq SO(3) \longrightarrow O_{\hat{L}_0}$. Note that $O_{\hat{L}_0} \simeq S^2$ for $\hat{L}_0 \neq 0$

and that the equation for $\Pi(t)$ within $O_{\hat{L}_0}$ reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\Pi(t) = \Pi(t) \times \Psi\left(R_M^{-1}\dot{R}_M\right) = \Pi(t) \times \Psi\left(I^{-1}_{d_0^{\mathrm{Mec}}(t)}\Pi(t) - Ad_{g^{-1}}\hat{\omega}_l(t)\right).$$

The reconstruction procedure follows the lines of [3] and Section 4.5. Suppose that the solution $\Pi(t)$ describes a closed simple curve on the sphere $S^2 = O_{\hat{L}_0}$, $\Pi(t_1) = \Pi(t_2) = \hat{L}_0$; then

$$R_M(t_2) = \exp\left((\theta^{\mathrm{Dyn}}(t_2) + \theta^{\mathrm{Geom}})\frac{\hat{L}_0}{\left\|\hat{L}_0\right\|}\right)$$

where the *geometric phase* angle θ^{Geom} is (mod 2π) minus the signed solid angle determined by the closed path $\Pi(t)$ on the sphere and the *dynamical phase* $\theta^{\mathrm{Dyn}}(t)$ is

$$\theta^{\mathrm{Dyn}}(t) = \frac{1}{\left\|\hat{L}_0\right\|}\int_{t_1}^{t}\mathrm{d}s\left(\left\langle\Pi(s), I^{-1}_{d_0^{\mathrm{Mec}}(s)}\Pi(s)\right\rangle - \left\langle\hat{J}(\dot{c}), \hat{\omega}_l(s)\right\rangle\right).$$

In the above expression, the first term gives $2K - 2K_{\mathrm{int}}$, where K represents the *rotational kinetic energy* (see Appendix A) and the second term is the *magnetic potential energy* of the system (see [5] pp. 230). Finally, the corresponding *phase formula* for the physical curve $c(t) \in Q$

$$c(t_2) = h_M(t_2)\cdot\exp\left(\theta^{\mathrm{Dyn}}(t_2)\frac{\hat{L}_0}{\|\hat{L}_0\|}\right)\cdot\exp\left(\theta^{\mathrm{Geom}}\frac{\hat{L}_0}{\|\hat{L}_0\|}\right)\cdot d_0^{\mathrm{Mec}}(t_2)$$

which determines the exact position of the system for the dynamically defined time t_2 in which the *body angular momentum* $\Pi(t)$ returns to its initial value. This is the *affine-constrained* (magnetic) version of the result obtained in [3].

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Appendix A. Kinetic energy

Here we derive an expression for the kinetic energy of the mechanical system on Q, in terms of the *controlled variables curve* $d_0(t)$ and the *group unknown* $g(t)$.

We shall assume that the kinetic energy of the underlying simple mechanical system on Q (with or without controls) is given by a G-invariant metric Riemannian on Q. This means that, if $k : TQ \otimes TQ \longrightarrow TQ$ denotes this metric, the *kinetic energy* reads:

$$K : TQ \longrightarrow \mathbb{R}$$
$$K(v_q) = \frac{1}{2}k_q(v_q, v_q).$$

Now, on our *controlled system*, the physical curve $c(t) \in Q$ is of form (1) and, then, the velocity $\dot{c}(t)$ is given by (3). Thus, the kinetic energy on the controlled curve becomes

$$K(\dot{c}(t)) = K_{\mathrm{int}}(t) + \frac{1}{2}\langle I_0(t)(\xi(t)), \xi(t)\rangle + \langle J_0(t), \xi(t)\rangle \tag{57}$$

where

$$K_{\mathrm{int}}(t) = \frac{1}{2}k_{d_{0(t)}}\left(\dot{d}_0(t), \dot{d}_0(t)\right)$$

shall be called the *internal* (or *gauge*) *kinetic energy* and $\xi(t) = g^{-1}\frac{\mathrm{d}}{\mathrm{d}t}g(t) \in \mathfrak{g}$, $I_0(t) = I_{d_0(t)}$, $J_0(t) := J(\dot{d}_0(t))$ as in Section 3.1.

In terms of the *body momentum* $\Pi(t)$ defined by Eq. (5), the expression takes the form:

$$K\left(\frac{\mathrm{d}}{\mathrm{d}t}c(t)\right) = K_{\text{int}}(t) + \frac{1}{2}\left\langle I_0^{-1}(t)(\Pi(t)), \Pi(t)\right\rangle - \frac{1}{2}\left\langle I_0^{-1}(t)(J_0(t)), J_0(t)\right\rangle$$

where the last term can be interpreted as a *gauge-dependent energy contribution* which appears because of the use of the "*moving reference system*" represented by $d_0(t)$ in Q.

Remark A.1 (*Mechanical Energy*). If there are also potential forces present in the mechanical system on Q, represented by a *potential* $V : Q \longrightarrow \mathbb{R}$, then the total *mechanical energy* is $E = K(\frac{\mathrm{d}}{\mathrm{d}t}c(t)) + V(g \cdot d_0(t))$. If V is G-invariant, then $E = K(\frac{\mathrm{d}}{\mathrm{d}t}c(t)) + V(d_0(t))$. Notice that as, in general, the *control forces* are non-potential and time-dependent, they do *work* on the system. So the above mechanical energy is *not conserved* during the controlled motion.

Remark A.2 (*Gauge Potential Interaction*). In terms of (1-d) gauge field theories, the term $\langle J_0(t), \xi(t)\rangle$ can be seen as a coupling between the *gauge field* J_0 and the *gauge variables* ξ (see also Remark 2.11).

Recall the *mechanical gauge* defined by (18). In this gauge, the kinetic energy is given by two *uncoupled* contributions:

$$K\left(\frac{\mathrm{d}}{\mathrm{d}t}c(t)\right) = K_{\text{int}}(t) + \frac{1}{2}\langle I_0(t)(\xi(t)), \xi(t)\rangle = K_{\text{int}}(t) + \frac{1}{2}\left\langle I_0^{-1}(t)(\Pi(t)), \Pi(t)\right\rangle.$$

Appendix B. Reconstruction on $G \longrightarrow O_\mu$

Consider the two maps [7] $T^*G \overset{\text{Body coord.}}{\leftrightarrow} G \times \mathfrak{g}^* \underset{\pi}{\overset{L}{\rightrightarrows}} \mathfrak{g}^*$, given by

$$L(g, \Pi) = Ad_g^* \Pi$$
$$\pi(g, \Pi) = \Pi$$

and suppose that we have a curve $(g(t), \Pi(t)) \in G \times \mathfrak{g}^*$ satisfying $L(g(t), \Pi(t)) = \mu = \text{const}$. The idea of this appendix is to reconstruct $g(t)$ from $\Pi(t)$ by means of the fact that $\Pi(t) = Ad^*_{g^{-1}(t)}\mu$.

Note that $\Pi(t)$ lies in the coadjoint orbit $O_\mu \subset \mathfrak{g}^*$ through μ. For reconstruction [6], we need to consider a *principal connection* on the G_μ-principal bundle $G \xrightarrow{\pi} O_\mu$, where $G_\mu := \{g \in G; Ad_g^*\mu = \mu\}$ denotes the stabilizer subgroup. This bundle corresponds to the reduction $L^{-1}(\mu) \approx G \xrightarrow{\pi} O_\mu$, where the G_μ action on $L^{-1}(\mu) \subset G \times \mathfrak{g}^*$ is the one induced by usual left action (in body coordinates) of G on T^*G. Using the principal bundle isomorphism

$$: L^{-1}(\mu) \xrightarrow{\approx} G$$
$$: (g, Ad^*_{g^{-1}}\mu) \longmapsto g$$

we see that a principal connection on $G \xrightarrow{\pi} O_\mu$ can be defined by a choice of a complement $HOR_e \subset \mathfrak{g}$ to the *isotropy Lie algebra* $\mathfrak{g}_\mu = \text{Lie}(G_\mu)$, i.e., $\mathfrak{g} = HOR_e \oplus \mathfrak{g}_\mu$, and by then *right translating* this complement to any point $g \in G$. There is no canonical way of choosing HOR_e in general. So, let $P : \mathfrak{g} \longrightarrow \mathfrak{g}_\mu$ be a linear projector onto $\mathfrak{g}_\mu$ such that

$$Ad_h \circ P = P \circ Ad_h \tag{58}$$

for all $h \in G_\mu$ and define $HOR_e = \text{Ker}(P)$. The corresponding *connection* 1-*form* $A_P : TG \longrightarrow \mathfrak{g}_\mu$ induced by P is then given by

$$A_P(v_g)_g := P(v_g g^{-1})$$

for $v_g \in T_g G$ and $v_g g^{-1}$ denoting the derivative at g of the right translation by g^{-1} in G.

Example B.1 (*Ad-invariant Metrics*). If the Lie algebra $\mathfrak{g}$ is equipped with an Ad-invariant scalar product $(,)$, then let P be the *orthogonal projector* with respect to $(,)$ onto $\mathfrak{g}_\mu$. It can be easily seen that this projector P satisfies (58), inducing a principal connection on $G \xrightarrow{\pi} O_\mu$.

Now, we shall make use of this connection to *reconstruct* $g(t)$ from a solution $\Pi(t)$ on the coadjoint orbit O_μ. Following [6]:

- consider the *horizontal lift* $g_G(t) \in G$ from $g_G(t_1) = g(t_1)$ of the base curve $\Pi(t) \in O_\mu$ with respect to the connection A_P,
- find $h_D(t)$ as the curve in G_μ fixed by requiring that

  $$g(t) = h_D(t) \cdot g_G(t)$$

  and thus yielding the desired solution of the reconstruction equation $\Pi(t) = Ad^*_{g^{-1}(t)}\mu$, for the initial value $g(t_1)$.

The group elements in the above decomposition of $g(t)$ at time t, $h_D(t)$ and $g_G(t)$, are usually called the *dynamic phase* and the *geometric phase*, respectively. The curve $h_D(t)$ must be a solution of

$$\frac{\mathrm{d}}{\mathrm{d}t} h_D h_D^{-1}(t) = A_P\left(\frac{\mathrm{d}}{\mathrm{d}t} g\right)_g \tag{59}$$

with $h_D(t_1) = e$.

Suppose now that $\mathfrak{g}$ has an Ad-invariant scalar product $(,)$ as in Example B.1. This bilinear form induces a vector space isomorphism $\Psi : \mathfrak{g}^* \longrightarrow \mathfrak{g}$ which transforms the coadjoint action into the adjoint action of G. Let $u_1 = \frac{\Psi(\mu)}{\|\Psi(\mu)\|}$ and $\{u_i\}_{i=1}^{\dim \mathfrak{g}_\mu}$ denote an orthonormal basis with respect to $(,)$ of the vector subspace $\mathfrak{g}_\mu \subset \mathfrak{g}$. Note that this can always be done since $\Psi(\mu) \in \mathfrak{g}_\mu$. The orthogonal projector, in this case, can be written as

$$P(v) = \Sigma_{i=1}^{\dim \mathfrak{g}_\mu} (u_i, v)\, u_i$$

and, thus,

$$\frac{\mathrm{d}}{\mathrm{d}t} h_D h_D^{-1}(t) = \Sigma_{i=1}^{\dim \mathfrak{g}_\mu} \left(u_i, \frac{\mathrm{d}}{\mathrm{d}t} g g^{-1}\right) u_i. \tag{60}$$

References

[1] R. Abraham, J. Marsden, Foundations of Mechanics, 2nd edition, Addison-Wesley, Menlo Park, California.

[2] A.M. Bloch, P.S. Krishnaprasad, J.E. Marsden, R. Murray, Nonholonomic mechanical systems with symmetry, Arch. Ration. Mech. Anal. 136, 21–99.

[3] A. Cabrera, A generalized Montgomery phase formula for rotating self-deforming bodies, J. Geom. Phys. 57 (2007) 1405–1420.

[4] H. Cendra, J.E. Marsden, T.S. Ratiu, Lagrangian reduction by stages, Mem. Amer. Math. Soc. 152 (2001) 722.

[5] H. Goldstein, Classical Mechanics, 2nd edition, Spanish version, Editorial Reverte S.A., 1992.

[6] J.E. Marsden, R. Montgomery, T. Ratiu, Reduction, symmetry and phases in mechanics, Mem. Amer. Math. Soc. 88 (436) (1990) 1–110.

[7] J.E. Marsden, T. Ratiu, Introduction to Mechanics and Symmetry, Springer-Verlag, 1994.

[8] J.E. Marsden, T. Ratiu, J. Scheurle, Reduction theory and the Lagrange–Routh equations, J. Math. Phys. 41 (2000) 3379.

[9] R. Montgomery, Optimal control of deformable bodies and its relation to gauge theory, in: T. Ratiu (Ed.), Proc. Geometry of Hamiltonian Systems Workshop, Springer, Berlin, 1988.

[10] R. Montgomery, How much does the rigid body rotate? A Berrys phase from the 18th century, Am. J. Phys. 59 (5) (1991) 394–398.

[11] A. Shapere, F. Wilczek, Gauge kinematics of deformable bodies, Am. J. Phys. 57 (6) (1989) 514–518.